

# A TWO-DIMENSIONAL UNIVOQUE SET

MARTIJN DE VRIES AND VILMOS KOMORNIK

**ABSTRACT.** Let  $\mathbf{J} \subset \mathbb{R}^2$  be the set of couples  $(x, q)$  with  $q > 1$  such that  $x$  has at least one representation of the form  $x = \sum_{i=1}^{\infty} c_i q^{-i}$  with integer coefficients  $c_i$  satisfying  $0 \leq c_i < q$ ,  $i \geq 1$ . In this case we say that  $(c_i) = c_1 c_2 \dots$  is an expansion of  $x$  in base  $q$ . Let  $\mathbf{U}$  be the set of couples  $(x, q) \in \mathbf{J}$  such that  $x$  has exactly one expansion in base  $q$ .

In this paper we deduce some topological and combinatorial properties of the set  $\mathbf{U}$ . We characterize the closure of  $\mathbf{U}$ , and we determine its Hausdorff dimension. For  $(x, q) \in \mathbf{J}$ , we also prove new properties of the lexicographically largest expansion of  $x$  in base  $q$ .

## 1. INTRODUCTION

Let  $\mathbf{J}$  be the set consisting of all elements  $(x, q) \in \mathbb{R} \times (1, \infty)$  such that there exists at least one sequence  $(c_i) = c_1 c_2 \dots$  of integers satisfying  $0 \leq c_i < q$  for all  $i$ , and

$$(1.1) \quad x = \frac{c_1}{q} + \frac{c_2}{q^2} + \dots.$$

If (1.1) holds, we say that  $(c_i)$  is an *expansion of  $x$  in base  $q$* , and if the base  $q$  is understood from the context, we sometimes simply say that  $(c_i)$  is an expansion of  $x$ . The numbers  $c_i$  of an expansion  $(c_i)$  are usually referred to as *digits*. We denote by  $\lceil q \rceil$  the smallest integer larger than or equal to  $q$ . The *alphabet*  $A_q$  is the set of “admissible” digits in base  $q$ , i.e.,  $A_q = \{0, \dots, \lceil q \rceil - 1\}$ .

If  $q > 1$  and  $0 \leq x \leq (\lceil q \rceil - 1)/(q - 1)$ , then a particular expansion of  $x$  in base  $q$ , the so-called *quasi-greedy expansion*  $(a_i(x, q))$ , may be defined recursively as follows. For  $x = 0$  we set  $(a_i(x, q)) := 0^\infty$ . If  $x > 0$  and  $a_i(x, q)$  has already been defined for  $1 \leq i < n$  (no condition if  $n = 1$ ), then  $a_n(x, q)$  is the largest element of  $A_q$  satisfying

$$\frac{a_1(x, q)}{q} + \dots + \frac{a_n(x, q)}{q^n} < x.$$

One easily verifies that  $(a_i(x, q))$  is indeed an expansion of  $x$  in base  $q$ . Therefore

$$(x, q) \in \mathbf{J} \iff q > 1 \quad \text{and} \quad x \in J_q := \left[0, \frac{\lceil q \rceil - 1}{q - 1}\right].$$

Let us denote by  $\mathbf{U}$  the set of couples  $(x, q) \in \mathbf{J}$  such that  $x$  has *exactly one* expansion in base  $q$ . For example,  $(0, q) \in \mathbf{U}$  for every  $q > 1$ , but  $\mathbf{U}$  has much more elements. The main purpose of this paper is to describe the topological and combinatorial nature of  $\mathbf{U}$ . We will prove the following theorem:

### Theorem 1.1.

- (i) *The set  $\mathbf{U}$  is not closed. Its closure  $\overline{\mathbf{U}}$  is a Cantor set*<sup>1</sup>.

*Date:* December 17, 2010.

*2000 Mathematics Subject Classification.* Primary:11A63, Secondary:11B83.

*Key words and phrases.* Greedy expansion, beta-expansion, univoque sequence, univoque set, Cantor set, Hausdorff dimension.

<sup>1</sup>We recall that a Cantor set is a nonempty closed set having neither interior nor isolated points.

- (ii) Both  $\mathbf{U}$  and  $\overline{\mathbf{U}}$  are two-dimensional Lebesgue null sets.
- (iii) Both  $\mathbf{U}$  and  $\overline{\mathbf{U}}$  have Hausdorff dimension two.

As far as we know this two-dimensional *univoque* set has not yet been investigated. There exists, however, a number of papers devoted to the study of its one-dimensional sections

$$\mathcal{U} := \{q > 1 : (1, q) \in \mathbf{U}\}$$

and

$$\mathcal{U}_q := \{x \in J_q : (x, q) \in \mathbf{U}\}, \quad q > 1.$$

The study of  $\mathcal{U}$  started with the paper of Erdős, Horváth and Joó [6] and was studied subsequently in [4], [5], [7], [8], [13], [14], [15]. We recall in particular that  $\mathcal{U}$  and its closure  $\overline{\mathcal{U}}$  have Lebesgue measure zero and Hausdorff dimension one.

The sets  $\mathcal{U}_q$  have been investigated in [3], [4], [5], [10], [11], [12]. It is known that  $\mathcal{U}_q$  is closed if and only if  $q$  does not belong to the null set  $\overline{\mathcal{U}}$ , and that its closure  $\overline{\mathcal{U}_q}$  has Lebesgue measure zero for all non-integer bases  $q > 1$ . Moreover, the set of numbers  $x \in J_q$  having continuum many expansions in base  $q$  has full Lebesgue measure for each non-integer  $q > 1$  (see [2], [18], [19]).

The key to the proof of Theorem 1.1 is an algebraic characterization of  $\overline{\mathbf{U}}$  by using the quasi-greedy expansions  $(a_i(x, q))$ . We write for brevity  $\alpha_i(q) := a_i(1, q)$ ,  $i \in \mathbb{N} := \{1, 2, \dots\}$ ,  $q > 1$ . Note that  $\alpha_1(q) = \lceil q \rceil - 1$ , the largest admissible digit in base  $q$ . In the statement of the following theorem we use the lexicographic order between sequences and we define the *conjugate* of the number  $a_i(x, q)$  by  $\overline{a_i(x, q)} := \alpha_1(q) - a_i(x, q)$ . If  $q > 1$  and  $c_i \in A_q$ ,  $i \geq 1$ , we shall also write  $\overline{c_1 \dots c_n}$  instead of  $\overline{c_1} \dots \overline{c_n}$  and  $\overline{c_1 c_2 \dots}$  instead of  $\overline{c_1} \overline{c_2} \dots$ .

**Theorem 1.2.** *A point  $(x, q) \in \mathbf{J}$  belongs to  $\overline{\mathbf{U}}$  if and only if*

$$\overline{a_{n+1}(x, q) a_{n+2}(x, q) \dots} \leq \alpha_1(q) \alpha_2(q) \dots \quad \text{whenever } a_n(x, q) > 0.$$

Along with the quasi-greedy expansion, we also need the notion of the *greedy expansion*  $(b_i(x, q))$  for  $x \in J_q$ , introduced by Rényi [17]. It can be defined by a slight modification of the above recursion: if  $b_i(x, q)$  has already been defined for all  $1 \leq i < n$  (no condition if  $n = 1$ ), then  $b_n(x, q)$  is the largest element of  $A_q$  satisfying

$$\frac{b_1(x, q)}{q} + \dots + \frac{b_n(x, q)}{q^n} \leq x.$$

Note that the greedy expansion  $(b_i(x, q))$  of a number  $x \in J_q$  is the lexicographically largest expansion of  $x$  in base  $q$ . We denote the greedy expansion of 1 in base  $q$  by  $(\beta_i(q)) := (b_i(1, q))$ .

The rest of this paper is organized as follows. In the next section we give a short overview of some basic results on greedy and quasi-greedy expansions, and we prove some new results concerning the coordinate-wise convergence of sequences of these expansions. We shall prove (see Theorem 2.7) that the set of numbers  $x \in J_q$  for which the greedy expansion of  $x$  in base  $q$  is not the greedy expansion of a number belonging to  $J_p$  in any smaller base  $p \in (1, q)$  is of full Lebesgue measure and its complement in  $J_q$  is a set of first category and Hausdorff dimension one. We shall also prove (see Theorem 2.8) that for each word  $v := b_{\ell+1}(x, q) \dots b_{\ell+m}(x, q)$  ( $\ell \geq 0, m \geq 1, x \in [0, 1)$ ) there exists a set  $Y_v \subset J_q$  of first category and Hausdorff dimension less than one, such that the word  $v$  occurs in the greedy expansion in base  $q$  of every number belonging to  $J_q \setminus Y_v$ . Using (some of) the results of Section 2 we prove Theorem 1.2 in Section 3 and Theorem 1.1 in Section 4.

## 2. GREEDY AND QUASI-GREEDY EXPANSIONS

In this paper we consider only one-sided sequences of nonnegative integers. We equip this set of sequences  $\{0, 1, \dots\}^{\mathbb{N}}$  with the topology of coordinate-wise convergence. We say that an expansion is *infinite* if it has infinitely many nonzero elements; otherwise it is called *finite*. Using this terminology, the quasi-greedy expansion  $(a_i(x, q))$  of a number  $x \in J_q \setminus \{0\}$  is the lexicographically largest *infinite* expansion of  $x$  in base  $q$ .

The family of all quasi-greedy expansions is characterized by the following propositions (see [1] or [5] for a proof):

**Proposition 2.1.** *The map  $q \mapsto (\alpha_i(q))$  is a strictly increasing bijection from the open interval  $(1, \infty)$  onto the set of all infinite sequences  $(\alpha_i)$  satisfying*

$$\alpha_{k+1}\alpha_{k+2}\dots \leq \alpha_1\alpha_2\dots \quad \text{for all } k \geq 1.$$

**Proposition 2.2.** *For each  $q > 1$ , the map  $x \mapsto (a_i(x, q))$  is a strictly increasing bijection from  $J_q \setminus \{0\}$  onto the set of all infinite sequences  $(a_i)$  satisfying*

$$a_n \in A_q \quad \text{for all } n \geq 1$$

and

$$a_{n+1}a_{n+2}\dots \leq \alpha_1(q)\alpha_2(q)\dots \quad \text{whenever } a_n < \alpha_1(q).$$

The quasi-greedy expansions have a lower semicontinuity property for the order topology induced by the lexicographic order. More precisely, we have the following result.

**Lemma 2.3.** *Let  $(x, q) \in \mathbf{J}$ . Then*

- (i) *for each positive integer  $m$  there exists a neighborhood  $\mathbf{W} \subset \mathbb{R}^2$  of  $(x, q)$  such that*

$$(2.1) \quad a_1(y, r) \dots a_m(y, r) \geq a_1(x, q) \dots a_m(x, q) \quad \text{for all } (y, r) \in \mathbf{W} \cap \mathbf{J};$$

- (ii) *if  $(y_n, r_n)$  converges to  $(x, q)$  in  $\mathbf{J}$  from below, then  $(a_i(y_n, r_n))$  converges to  $(a_i(x, q))$ .*

*Proof.* (i) We may assume that  $x \neq 0$ . By definition of the quasi-greedy expansion we have

$$\sum_{i=1}^n \frac{a_i(x, q)}{q^i} < x \quad \text{for all } n = 1, 2, \dots$$

For any fixed positive integer  $m$ , if  $(y, r) \in \mathbf{J}$  is sufficiently close to  $(x, q)$ , then  $r > [q] - 1$ , i.e.,  $A_q \subset A_r$ , and

$$\sum_{i=1}^n \frac{a_i(x, q)}{r^i} < y, \quad n = 1, \dots, m.$$

These inequalities imply (2.1).

(ii) If  $y_n \leq x$  and  $r_n \leq q$ , we deduce from the definition of the quasi-greedy expansion that

$$(a_i(x, q)) \geq (a_i(y_n, r_n))$$

for every  $n$ . Equivalently, we have

$$a_1(x, q) \dots a_m(x, q) \geq a_1(y_n, r_n) \dots a_m(y_n, r_n)$$

for all positive integers  $m$  and  $n$ . It remains to notice that by the previous part the converse inequality also holds for each fixed  $m$  if  $n$  is large enough.  $\square$

The family of greedy expansions has already been characterized by Parry [16]:

**Proposition 2.4.** *For a given base  $q > 1$ , the map  $x \mapsto (b_i(x, q))$  is a strictly increasing bijection from  $J_q$  onto the set of all sequences  $(b_i)$  satisfying*

$$b_n \in A_q \quad \text{for all } n \geq 1$$

and

$$b_{n+1}b_{n+2}\dots < \alpha_1(q)\alpha_2(q)\dots \quad \text{whenever } b_n < \alpha_1(q).$$

The greedy expansions have the following upper semicontinuity property:

**Lemma 2.5.** *Let  $(x, q) \in \mathbf{J}$  and suppose  $q$  is a non-integer. Then*

- (i) *for each positive integer  $m$  there exists a neighborhood  $\mathbf{W} \subset \mathbb{R}^2$  of  $(x, q)$  such that*
- (2.2) 
$$b_1(y, r) \dots b_m(y, r) \leq b_1(x, q) \dots b_m(x, q) \quad \text{for all } (y, r) \in \mathbf{W} \cap \mathbf{J};$$
- (ii) *if  $(y_n, r_n)$  converges to  $(x, q)$  in  $\mathbf{J}$  from above, then  $(b_i(y_n, r_n))$  converges to  $(b_i(x, q))$ .*

*Proof.* (i) By the definition of greedy expansions we have

$$\sum_{i=1}^n \frac{b_i(x, q)}{q^i} > x - \frac{1}{q^n} \quad \text{whenever } b_n(x, q) < \alpha_1(q).$$

If  $(y, r) \in \mathbf{J}$  is sufficiently close to  $(x, q)$ , then  $A_q = A_r$ ,  $\alpha_1(r) = \alpha_1(q)$ , and

$$\sum_{i=1}^n \frac{b_i(x, q)}{r^i} > y - \frac{1}{r^n} \quad \text{whenever } n \leq m \text{ and } b_n(x, q) < \alpha_1(r).$$

These inequalities imply (2.2).

(ii) If  $y_n \geq x$  and  $r_n \geq q$ , we deduce from the definition of the greedy expansion that

$$(b_i(x, q)) \leq (b_i(y_n, r_n))$$

for every  $n$ . Equivalently, we have

$$b_1(x, q) \dots b_m(x, q) \leq b_1(y_n, r_n) \dots b_m(y_n, r_n)$$

for all positive integers  $m$  and  $n$ . It remains to notice that by the previous part the converse inequality also holds for each fixed  $m$  if  $n$  is large enough.  $\square$

From Lemmas 2.3 and 2.5 we deduce the following result:

**Proposition 2.6.** *Consider  $(x, q) \in \mathbf{J}$  with a non-integer base  $q$  and assume that the greedy expansion  $(b_i(x, q))$  is infinite. If  $(y_n, r_n)$  converges to  $(x, q)$  in  $\mathbf{J}$ , then both  $(a_i(y_n, r_n))$  and  $(b_i(y_n, r_n))$  converge to  $(b_i(x, q)) = (a_i(x, q))$ .*

*Proof.* For each positive integer  $m$  there exists a neighborhood  $\mathbf{W} \subset \mathbb{R}^2$  of  $(x, q)$  such that for all  $(y, r) \in \mathbf{W} \cap \mathbf{J}$ ,

$$\begin{aligned} a_1(x, q) \dots a_m(x, q) &\leq a_1(y, r) \dots a_m(y, r) \\ &\leq b_1(y, r) \dots b_m(y, r) \\ &\leq b_1(x, q) \dots b_m(x, q). \end{aligned}$$

The result follows from our assumption that  $(a_i(x, q)) = (b_i(x, q))$ .  $\square$

**Theorem 2.7.** *Let  $q > 1$  be a real number. Then*

- (i) *for each  $r \in (1, q)$ , the Hausdorff dimension of the set*

$$G_{r,q} := \left\{ \sum_{i=1}^{\infty} \frac{b_i(x, r)}{q^i} : x \in J_r \right\}$$

*equals  $\log r / \log q$ ;*

(ii) the set

$$G_q := \bigcup \{G_{r,q} : r \in (1, q)\}$$

is of first category, has Lebesgue measure zero and Hausdorff dimension one.

*Proof.* (i) It is well known ([15], [16]) and easy to prove that the set of numbers  $r > 1$  for which  $(\beta_i(r))$  is finite is dense in  $[1, \infty)$ . Moreover, if  $(\beta_i(r))$  is finite and  $\beta_n(r)$  is its last nonzero element, then  $(\alpha_i(r)) = (\beta_1(r) \dots \beta_{n-1}(r) \beta_n^-(r))^\infty$  ( $\beta_n^-(r) := \beta_n(r) - 1$ ). By virtue of Propositions 2.1 and 2.4 we have  $G_{s,q} \subset G_{t,q}$  whenever  $1 < s < t < q$ . Hence it is enough to prove that  $\dim_H G_{r,q} = \log r / \log q$  for those values  $r \in (1, q)$  for which  $(\alpha_i(r))$  is periodic.

Fix  $r \in (1, q)$  such that  $(\alpha_i) := (\alpha_i(r))$  is periodic and let  $n \in \mathbb{N}$  be such that  $(\alpha_i) = (\alpha_1 \dots \alpha_n)^\infty$ . Let us denote by  $W_r$  the set consisting of the finite words

$$w_{ij} := \alpha_1 \dots \alpha_{j-1} i, \quad 0 \leq i < \alpha_j, \quad 1 \leq j \leq n$$

and

$$w_{\alpha_n n} := \alpha_1 \dots \alpha_{n-1} \alpha_n.$$

Let  $\mathcal{F}'_r$  be the set of sequences  $(c_i) = c_1 c_2 \dots$  such that for each  $k \geq 0$  the inequality  $c_{k+1} \dots c_{k+n} \leq \alpha_1 \dots \alpha_n$  holds. Note that the set  $\mathcal{F}'_r$  consists of those sequences  $(c_i)$  such that each tail of  $(c_i)$  (including  $(c_i)$  itself) starts with a word belonging to  $W_r$ . It follows from Propositions 2.1 and 2.4 that a sequence  $(b_i)$  is greedy in base  $r$  if and only if  $b_m \in A_r$  for all  $m \geq 1$  and

$$b_{m+k+1} b_{m+k+2} \dots < \alpha_1 \alpha_2 \dots \quad \text{for all } k \geq 0, \text{ whenever } b_m < \alpha_1.$$

Therefore, any greedy expansion  $(b_i) \neq \alpha_1^\infty$  in base  $r$  can be written as  $\alpha_1^\ell c_1 c_2 \dots$  for some  $\ell \geq 0$  ( $\alpha_1^0$  denotes the empty word) and some sequence  $(c_i)$  belonging to  $\mathcal{F}'_r$ . Conversely, if no tail of a sequence belonging to  $\mathcal{F}'_r$  equals  $(\alpha_i)$ , then it is the greedy expansion in base  $r$  of some  $x \in J_r$ . Hence if we set

$$\mathcal{F}_{r,q} := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{q^i} : (c_i) \in \mathcal{F}'_r \right\},$$

then  $\mathcal{F}_{r,q} \setminus G_{r,q}$  is countable and  $G_{r,q}$  can be covered by countably many sets similar to  $\mathcal{F}_{r,q}$ . Since the union of countably many sets of Hausdorff dimension  $s$  is still of Hausdorff dimension  $s$ , we have  $\dim_H G_{r,q} = \dim_H \mathcal{F}_{r,q}$ .

We associate with each word  $w_{ij} \in W_r$  a similarity  $S_{ij} : J_q \rightarrow J_q$  defined by the formula

$$S_{ij}(x) := \frac{\alpha_1}{q} + \dots + \frac{\alpha_{j-1}}{q^{j-1}} + \frac{i}{q^j} + \frac{x}{q^j}, \quad x \in J_q.$$

It follows from Proposition 2.1 and the definition of  $\mathcal{F}_{r,q}$  that

$$(2.3) \quad \mathcal{F}_{r,q} = \bigcup S_{ij}(\mathcal{F}_{r,q})$$

where the union runs over all  $i$  and  $j$  for which  $w_{ij} \in W_r$ . Applying Proposition 2.1 again, it follows that  $r$  is the largest element of the set of numbers  $t > 1$  for which  $\alpha_i(t) = \alpha_i$ ,  $1 \leq i \leq n$ . Hence  $\alpha_1 \dots \alpha_n < \alpha_1(q) \dots \alpha_n(q)$  and therefore each sequence in  $\mathcal{F}'_r$  is the greedy expansion in base  $q$  of some  $x \in \mathcal{F}_{r,q}$ . It follows that the sets  $S_{ij}(\mathcal{F}_{r,q})$  on the right side of (2.3) are disjoint. Moreover, the function  $x \mapsto (b_i(x, q))$  that maps  $\mathcal{F}_{r,q}$  onto  $\mathcal{F}'_r$  is increasing. Using the definition of  $\mathcal{F}'_r$  it is easily seen that the limit of each monotonic sequence of elements in  $\mathcal{F}_{r,q}$  belongs to  $\mathcal{F}_{r,q}$ . We conclude that the closed set  $\mathcal{F}_{r,q}$  is the (nonempty compact) invariant set of this system of similarities. An application of Propositions 9.6 and 9.7 in [9] yields that

$$\dim_H \mathcal{F}_{r,q} = \dim_H G_{r,q} = s$$

where  $s$  is the real solution of the equation

$$\frac{\alpha_1}{q^s} + \dots + \frac{\alpha_{n-1}}{q^{(n-1)s}} + \frac{\alpha_n + 1}{q^{ns}} = 1.$$

Since

$$\frac{\alpha_1}{r} + \dots + \frac{\alpha_{n-1}}{r^{n-1}} + \frac{\alpha_n + 1}{r^n} = 1$$

we have  $s = \log r / \log q$ .

(ii) It follows at once from Theorem 2.7(i) that  $\dim_H G_q = 1$ . Let  $r \in (1, q)$  be such that  $(\alpha_i(r))$  is periodic. The proof of Theorem 2.7(i) shows that

$$G_{r,q} \subset \bigcup_{n=1}^{\infty} (a_n + b_n \mathcal{F}_{r,q})$$

for some constants  $a_n, b_n \in \mathbb{R}$  ( $n \in \mathbb{N}$ ). Since  $\mathcal{F}_{r,q}$  is a closed set of Hausdorff dimension less than one, it follows in particular that the sets  $a_n + b_n \mathcal{F}_{r,q}$  are nowhere dense null sets. Since  $G_{s,q} \subset G_{t,q}$  whenever  $1 < s < t < q$ , the set  $G_q$  is a null set of first category.  $\square$

**Theorem 2.8.** *Let  $q > 1$  be a real number.*

- (i) *Let  $v := b_{\ell+1}(y, q) \dots b_{\ell+m}(y, q)$  for some  $y \in [0, 1)$  and some integers  $\ell \geq 0$  and  $m \geq 1$ . The set  $Y_v$  of numbers  $x \in J_q$  for which the word  $v$  does not occur in the greedy expansion of  $x$  in base  $q$  has Hausdorff dimension less than one.*
- (ii) *The set  $Y$  of numbers  $x \in J_q$  for which at least one word of the form  $b_{\ell+1}(y, q) \dots b_{\ell+m}(y, q)$  ( $\ell \geq 0, m \geq 1, y \in [0, 1)$ ) does not occur in the greedy expansion of  $x$  in base  $q$  is of first category, has Lebesgue measure zero and Hausdorff dimension one.*

*Proof.* (i) Using the inequality  $(b_i(y, q)) < (\alpha_i(q))$ , it follows from Proposition 2.4 that for some  $k \in \mathbb{N}$ , there exist positive integers  $m_1, \dots, m_k$  and nonnegative integers  $\ell_1, \dots, \ell_k$  satisfying  $\alpha_{m_j}(q) > 0$  and  $\ell_j < \alpha_{m_j}(q)$  for each  $1 \leq j \leq k$ , such that  $v$  is a subword of

$$w := \alpha_1(q) \dots \alpha_{m_1-1}(q) \ell_1 \dots \alpha_1(q) \dots \alpha_{m_k-1}(q) \ell_k.$$

Let  $W_q$  and  $\mathcal{F}'_q$  be the same as the sets  $W_r$  and  $\mathcal{F}'_r$  defined in the proof of the previous theorem, but now with  $(\alpha_i) := (\alpha_i(q))$  and  $n \geq \max\{m_1, \dots, m_k\}$  large enough such that the inequality

$$(2.4) \quad \left(1 + \frac{1}{q^n}\right)^k < 1 + \frac{1}{q^{m_1 + \dots + m_k}}$$

holds. If  $w_{i_1 j_1}, \dots, w_{i_k j_k}$  are  $k$  words belonging to  $W_q$  such that

$$i_1 j_1 \dots i_k j_k \neq \ell_1 m_1 \dots \ell_k m_k,$$

we associate with them a similarity  $S_{i_1 j_1 \dots i_k j_k} : J_q \rightarrow J_q$  defined by the formula

$$\begin{aligned} S_{i_1 j_1 \dots i_k j_k}(x) &= \frac{\alpha_1}{q} + \dots + \frac{\alpha_{j_1-1}}{q^{j_1-1}} + \frac{i_1}{q^{j_1}} \\ &\quad + \frac{\alpha_1}{q^{j_1+1}} + \dots + \frac{\alpha_{j_2-1}}{q^{j_1+j_2-1}} + \frac{i_2}{q^{j_1+j_2}} \\ &\quad \vdots \\ &\quad + \frac{\alpha_1}{q^{j_1+\dots+j_{k-1}+1}} + \dots + \frac{\alpha_{j_k-1}}{q^{j_1+\dots+j_k-1}} + \frac{i_k}{q^{j_1+\dots+j_k}} \\ &\quad + \frac{x}{q^{j_1+\dots+j_k}}, \quad x \in J_q. \end{aligned}$$

Let  $\mathcal{G}'_q$  denote the set of those sequences belonging to  $\mathcal{F}'_q$  which do not contain the word  $w$ , and let

$$\mathcal{G}_q := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{q^i} : (c_i) \in \mathcal{G}'_q \right\}.$$

Since  $(\alpha_i) = (\alpha_i(q))$ , a sequence belonging to  $\mathcal{F}'_q$  is not necessarily the greedy expansion in base  $q$  of a number  $x \in J_q$ , but this does not affect our proof. It is important, however, that any greedy expansion  $(b_i) \neq \alpha_1^\infty$  in base  $q$  can be written as  $\alpha_1^\ell c_1 c_2 \dots$  for some  $\ell \geq 0$  and some sequence  $(c_i)$  belonging to  $\mathcal{F}'_q$ . If  $Y_w$  denotes the set of numbers  $x \in J_q$  for which the word  $w$  does not occur in  $(b_i(x, q))$  then the latter fact implies that the set  $Y_w \setminus \{\alpha_1/(q-1)\}$  can be covered by countably many sets similar to  $\mathcal{G}_q$ .

It follows from the definition of  $\mathcal{G}_q$  that

$$\mathcal{G}_q \subset \bigcup S_{i_1 j_1 \dots i_k j_k}(\mathcal{G}_q)$$

where the union runs over all  $i_1 j_1 \dots i_k j_k$  for which the similarity  $S_{i_1 j_1 \dots i_k j_k}$  is defined above. Hence

$$\overline{\mathcal{G}_q} \subset \bigcup S_{i_1 j_1 \dots i_k j_k}(\overline{\mathcal{G}_q})$$

and thus  $\mathcal{G}_q \subset \mathcal{H}_q$  where  $\mathcal{H}_q$  is the (nonempty compact) invariant set of this system of similarities. Let  $\tilde{\alpha}_i := \alpha_i$  for  $1 \leq i < n$  and  $\tilde{\alpha}_n := \alpha_n + 1$ . From Proposition 9.6 in [9] we know that  $\dim_H \mathcal{H}_q \leq s$  where  $s$  is the real solution of the equation

$$(2.5) \quad \sum_{j_1=1}^n \sum_{j_2=1}^n \dots \sum_{j_k=1}^n \left( \frac{\prod_{i=1}^k \tilde{\alpha}_{j_i}}{q^{(j_1+\dots+j_k)s}} \right) - \frac{1}{q^{(m_1+\dots+m_k)s}} = 1.$$

Denoting the left side of (2.5) by  $C(s)$ , we have

$$C(1) + \frac{1}{q^{m_1+\dots+m_k}} = \left( \sum_{i=1}^n \frac{\tilde{\alpha}_i}{q^i} \right)^k < \left( 1 + \frac{1}{q^n} \right)^k.$$

By (2.4) we have  $C(1) < 1$ , and thus  $\dim_H Y_v \leq \dim_H Y_w \leq \dim_H \mathcal{H}_q < 1$ .

(ii) An argument analogous to the one used in the proof of Theorem 2.7(ii) shows that the set  $Y_w$  (and thus  $Y_v$ ) is of first category. Since  $Y$  is a countable union of sets of the form  $Y_v$  it follows that  $Y$  is a null set of first category. Let  $r \in (1, q)$  and let  $G_{r,q}$  be the set defined in Theorem 2.7. Due to Theorem 2.7(i) it is now sufficient to show that  $G_{r,q} \subset Y$ . To this end, choose a number  $p \in (r, q)$ , and let  $n \in \mathbb{N}$  be large enough such that the inequalities

$$\alpha_1(r) \dots \alpha_n(r) < \alpha_1(p) \dots \alpha_n(p) < \alpha_1(q) \dots \alpha_n(q)$$

hold. Note that such an integer  $n$  exists by Proposition 2.1. From Propositions 2.1 and 2.4 we conclude that the sequence  $0\alpha_1(p) \dots \alpha_n(p)0^\infty$  equals  $(b_i(y, q))$  for some  $y \in [0, 1)$  while the word  $0\alpha_1(p) \dots \alpha_n(p)$  does not occur in the greedy expansion in base  $r$  of any number  $x \in J_r$ .  $\square$

### 3. PROOF OF THEOREM 1.2

The following characterization of unique expansions readily follows from Proposition 2.4.

**Proposition 3.1.** *Fix  $q > 1$ . A sequence  $(c_i)$  of integers  $c_i \in A_q$  is the unique expansion of some  $x \in J_q$  if and only if*

$$c_{n+1}c_{n+2} \dots < \alpha_1(q)\alpha_2(q) \dots \quad \text{whenever } c_n < \alpha_1(q)$$

and

$$\overline{c_{n+1}c_{n+2} \dots} < \alpha_1(q)\alpha_2(q) \dots \quad \text{whenever } c_n > 0.$$

In what follows we use the notations  $(a_i(x, q))$ ,  $(b_i(x, q))$ ,  $(\alpha_i(q))$  and  $(\beta_i(q))$  as introduced in Section 1. If  $x$  and  $q$  are clear from the context, then we omit these arguments and we simply write  $a_i$ ,  $b_i$ ,  $\alpha_i$  and  $\beta_i$ . If two couples  $(x, q)$  and  $(x', q')$  are considered simultaneously, then we also write  $a'_i$ ,  $b'_i$ ,  $\alpha'_i$  and  $\beta'_i$  instead of  $a_i(x', q')$ ,  $b_i(x', q')$ ,  $\alpha_i(q')$  and  $\beta_i(q')$ .

**Lemma 3.2.** *Given  $(x, q) \in \mathbf{J}$ , the following two conditions are equivalent:*

$$\begin{aligned} \overline{a_{n+1}a_{n+2}\dots} &\leq \alpha_1\alpha_2\dots && \text{whenever } a_n > 0; \\ \overline{a_{n+1}a_{n+2}\dots} &\leq \beta_1\beta_2\dots && \text{whenever } a_n > 0. \end{aligned}$$

*Proof.* Since  $(\alpha_i) \leq (\beta_i)$ , it suffices to show that if there exists a positive integer  $n$  such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}a_{n+2}\dots} > \alpha_1\alpha_2\dots,$$

then there exists also a positive integer  $n'$  such that

$$a_{n'} > 0 \quad \text{and} \quad \overline{a_{n'+1}a_{n'+2}\dots} > \beta_1\beta_2\dots$$

If the greedy expansion  $(\beta_i)$  is infinite, then  $(\beta_i) = (\alpha_i)$  and we may choose  $n' = n$ . If  $(\beta_i)$  has a last nonzero digit  $\beta_\ell$ , then  $(\alpha_i) = (\alpha_1 \dots \alpha_\ell)^\infty$  with  $\alpha_1 \dots \alpha_{\ell-1}\alpha_\ell = \beta_1 \dots \beta_{\ell-1}\beta_\ell^-$  ( $\beta_\ell^- := \beta_\ell - 1$ ), and thus  $\alpha_\ell < \alpha_1$ . Since we have

$$\overline{a_{n+1}a_{n+2}\dots} > (\alpha_1 \dots \alpha_\ell)^\infty$$

by assumption, there exists a nonnegative integer  $j$  satisfying

$$\overline{a_{n+1}\dots a_{n+j\ell}} = (\alpha_1 \dots \alpha_\ell)^j \quad \text{and} \quad \overline{a_{n+j\ell+1}\dots a_{n+(j+1)\ell}} > \alpha_1 \dots \alpha_\ell.$$

Putting  $n' := n + j\ell$  it follows that

$$a_{n'} > 0 \quad \text{and} \quad \overline{a_{n'+1}\dots a_{n'+\ell}} \geq \beta_1 \dots \beta_\ell.$$

It follows from our assumption  $\overline{a_{n+1}a_{n+2}\dots} > \alpha_1\alpha_2\dots$  that  $(\alpha_i) < \alpha_1^\infty$  and  $(a_i) \neq \alpha_1^\infty$ . It follows from Proposition 2.2 that  $(a_i)$  has no tail equal to  $\alpha_1^\infty$ , so that  $\overline{a_{n'+\ell+1}a_{n'+\ell+2}\dots} > 0^\infty$ . We conclude that

$$\overline{a_{n'+1}a_{n'+2}\dots} > \beta_1\beta_2\dots \quad \square$$

*Definition.* We say that  $(x, q) \in \mathbf{J}$  belongs to the set  $\mathbf{V}$  if one of the two equivalent conditions of the preceding lemma is satisfied. Moreover, we define

$$\mathcal{V}_q := \{x \in J_q : (x, q) \in \mathbf{V}\}, \quad q > 1.$$

It follows from Proposition 3.1 that  $\mathbf{U} \subset \mathbf{V} \subset \mathbf{J}$ .

*Proof of Theorem 1.2.* We need to prove that  $\overline{\mathbf{U}} \cap \mathbf{J} = \mathbf{V}$ .

First we show that  $\mathbf{V} \subset \overline{\mathbf{U}}$ . In order to do so, we introduce for each fixed  $q > 1$  the sets  $\mathcal{U}'_q$  and  $\mathcal{V}'_q$ , defined by

$$\mathcal{U}'_q := \{(a_i(x, q)) : x \in \mathcal{U}_q\} \quad \text{and} \quad \mathcal{V}'_q := \{(a_i(x, q)) : x \in \mathcal{V}_q\}.$$

Observe that  $\mathcal{U}'_q$  is simply the set of unique expansions in base  $q$ . It follows easily from Propositions 2.1, 2.2 and 3.1 that  $\mathcal{U}'_q \subset \mathcal{V}'_q$  for each  $q > 1$ , and that  $\mathcal{V}'_q \subset \mathcal{U}'_r$  for each  $r > q$  such that  $\lceil q \rceil = \lceil r \rceil$ . Since we also have  $\overline{\mathcal{U}}_q = \mathcal{V}_q = [0, 1]$  if  $q > 1$  is an integer, the result follows.

Next we show that  $\overline{\mathbf{U}} \cap \mathbf{J} \subset \mathbf{V}$ . Since  $\mathbf{U} \subset \mathbf{V}$  it is sufficient to prove that if  $(x, q) \in \mathbf{J} \setminus \mathbf{V}$ , then  $(x', q') \notin \mathbf{V}$  for all  $(x', q') \in \mathbf{J}$  close enough to  $(x, q)$ . Applying Lemma 3.2 there exist two positive integers  $n$  and  $m$  such that

$$(3.1) \quad a_n > 0 \quad \text{and} \quad \overline{a_{n+1}\dots a_{n+m}} > \beta_1 \dots \beta_m.$$



This implies in particular that  $q$  is not an integer, because otherwise  $(\alpha_i) = (\beta_i) = \beta_1^\infty$ . Hence, if  $q'$  is sufficiently close to  $q$ , then

$$(3.2) \quad \beta'_1 \dots \beta'_m \leq \beta_1 \dots \beta_m$$

by Lemma 2.5. It follows from the definition of quasi-greedy expansions that

$$\frac{a_1}{q} + \dots + \frac{a_{j-1}}{q^{j-1}} + \frac{a_j^+}{q^j} + \frac{1}{q^{j+m}} > x \quad \text{whenever } a_j < \alpha_1,$$

where  $a_j^+ := a_j + 1$ . If  $(x', q') \in \mathbf{J}$  is sufficiently close to  $(x, q)$ , then  $\alpha_1 = \alpha'_1$ , the inequality (3.2) is satisfied,  $a'_1 \dots a'_{n+m} \geq a_1 \dots a_{n+m}$  by Lemma 2.3(i), and

$$(3.3) \quad \frac{a_1}{q'} + \dots + \frac{a_{j-1}}{(q')^{j-1}} + \frac{a_j^+}{(q')^j} + \frac{1}{(q')^{j+m}} > x' \quad \text{whenever } j \leq n+m \text{ and } a_j < \alpha_1.$$

Now we distinguish between two cases.

If  $a'_1 \dots a'_{n+m} = a_1 \dots a_{n+m}$ , then we have

$$a'_n > 0 \quad \text{and} \quad \overline{a'_{n+1} \dots a'_{n+m}} > \beta_1 \dots \beta_m \geq \beta'_1 \dots \beta'_m$$

by (3.1) and (3.2). This proves that  $(x', q') \notin \mathbf{V}$ .

If  $a'_1 \dots a'_{n+m} > a_1 \dots a_{n+m}$ , then let us consider the smallest  $j$  for which  $a'_j > a_j$ . It follows from (3.2) and (3.3) that

$$a'_j = a_j^+ > 0 \quad \text{and} \quad \overline{a'_{j+1} \dots a'_{j+m}} = \beta_1^m > \beta_1 \dots \beta_m \geq \beta'_1 \dots \beta'_m.$$

Hence  $(x', q') \notin \mathbf{V}$  again.  $\square$

*Remark.* It is the purpose of this remark to describe the set  $\overline{\mathbf{U}} \setminus \mathbf{J}$ . For each  $m \in \mathbb{N}$ , we define the number  $q_m \in (m, m+1)$  by the equation

$$1 = \frac{m}{q_m} + \frac{1}{q_m^2}.$$

Fix  $q \in (m, q_m]$ . Since  $\alpha_1(q) = m$  and  $\alpha_2(q) = 0$ , Proposition 3.1 implies that a sequence  $(c_i) \in \{0, \dots, m\}^{\mathbb{N}}$  belongs to  $\mathcal{U}'_q$  if and only if for each  $n \in \mathbb{N}$ , we have

$$c_n < m \implies c_{n+1} < m$$

and

$$c_n > 0 \implies c_{n+1} > 0.$$

Denoting the set of all such sequences by  $D'_m$  and putting for  $m > 1$  (note that  $D'_1 = \{0^\infty, 1^\infty\}$ ),

$$D_m := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{m^i} : (c_i) \in D'_m \right\},$$

one may verify that

$$\overline{\mathbf{U}} \setminus \mathbf{J} = \{(0, 1)\} \cup \bigcup_{m=2}^{\infty} (D_m \setminus [0, 1]) \times \{m\}.$$

#### 4. PROOF OF THEOREM 1.1

We need some results on the Hausdorff dimension of the sets  $\mathcal{U}_q$  and  $\mathcal{V}_q$  for  $q > 1$ . It follows from Theorem 1.2 that  $\mathcal{U}_q \subset \overline{\mathcal{U}_q} \subset \mathcal{V}_q$ . Moreover, if an element  $x \in \mathcal{V}_q \setminus \mathcal{U}_q$  has an infinite greedy expansion in base  $q$ , then  $(b_i(x, q))$  must end with  $\alpha_1(q)\alpha_2(q)\dots$  as follows from Propositions 2.4 and 3.1; hence  $\mathcal{V}_q \setminus \mathcal{U}_q$  is countable and the sets  $\mathcal{U}_q$ ,  $\overline{\mathcal{U}_q}$  and  $\mathcal{V}_q$  have the same Hausdorff dimension for each  $q > 1$ . Proposition 4.1 below is contained in the works of Daróczy and Kátai [4], Kallós [11], [12], Glendinning and Sidorov [10], and Sidorov [19]; for the reader's convenience we provide here an elementary proof.

**Proposition 4.1.** *We have*

- (i)  $\lim_{q \uparrow 2} \dim_H \mathcal{U}_q = 1$ ;
- (ii)  $\dim_H \mathcal{U}_q < 1$  for all non-integer  $q > 1$ .

*Proof.* (i) Assume that  $q \in (1, 2)$  is larger than the tribonacci number, i.e.,

$$\frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} < 1,$$

and let  $N = N(q) \geq 2$  be the largest integer satisfying

$$\frac{1}{q} + \cdots + \frac{1}{q^{2N-1}} < 1.$$

Hence  $\alpha_1(q) = \cdots = \alpha_{2N-1}(q) = 1$ . Let us denote by  $\mathcal{I}_q$  the set of numbers  $x \in J_q$  which have an expansion  $(c_i)$  satisfying  $0 < c_{kN+1} + \cdots + c_{kN+N} < N$  for every nonnegative integer  $k$ . Since in such expansions  $(c_i)$ , a zero (one) is followed by at most  $2N - 2$  consecutive one (zero) digits, it follows from Proposition 3.1 that  $\mathcal{I}_q \subset \mathcal{U}_q$ . Moreover, the set  $\mathcal{I}_q$  is closed and thus compact. In order to prove this, observe first that the set  $\mathcal{I}_q$  is closed from above <sup>2</sup> by virtue of Lemma 2.5(ii). Hence  $\mathcal{I}_q$  is closed because  $\mathcal{I}_q$  is symmetric relative to  $J_q$ .

It suffices now to prove that

$$(4.1) \quad \dim_H \mathcal{I}_q = \frac{\log(2^N - 2)}{N \log q};$$

indeed,  $q \uparrow 2$  implies  $N \rightarrow \infty$ , hence  $\dim_H \mathcal{I}_q \rightarrow 1$  and consequently  $\dim_H \mathcal{U}_q \rightarrow 1$ .

Observe that

$$(4.2) \quad \mathcal{I}_q = \bigcup S_{c_1 \dots c_N}(\mathcal{I}_q)$$

where the union runs over the words  $c_1 \dots c_N$  of length  $N$  consisting of zeros and ones satisfying  $0 < c_1 + \cdots + c_N < N$ , and  $S_{c_1 \dots c_N} : J_q \rightarrow J_q$  is given by

$$S_{c_1 \dots c_N}(x) := \left( \frac{c_1}{q} + \cdots + \frac{c_N}{q^N} \right) + \frac{x}{q^N}, \quad x \in J_q.$$

In other words,  $\mathcal{I}_q$  is the (nonempty compact) invariant set of the iterated function system formed by these  $2^N - 2$  similarities. The sets  $S_{c_1 \dots c_N}(\mathcal{I}_q)$  on the right side of (4.2) are disjoint because  $S_{c_1 \dots c_N}(\mathcal{I}_q) \subset \mathcal{I}_q \subset \mathcal{U}_q$ , and since all similarity ratios are equal to  $q^{-N}$ , it follows from Propositions 9.6 and 9.7 in [9] that the Hausdorff dimension  $s$  of  $\mathcal{I}_q$  is the real solution of the equation

$$(2^N - 2)q^{-Ns} = 1,$$

which is equivalent to (4.1).

(ii) Let  $q > 1$  be a non-integer and let  $n \in \mathbb{N}$  be such that  $\alpha_n(q) < \alpha_1(q)$ . It follows from Proposition 3.1 that the word  $1(0)^n$  does not occur in  $(b_i(x, q))$  if  $x$  belongs to  $\mathcal{U}_q$ . Applying Theorem 2.8(i) with  $y = q^{-1}$ ,  $\ell = 0$  and  $m = n + 1$ , we conclude that  $\dim_H \mathcal{U}_q < 1$ .  $\square$

*Proof of Theorem 1.1.* (ii) Let  $q > 1$  be a non-integer. Since  $\mathcal{V}_q \setminus \mathcal{U}_q$  is countable, Proposition 4.1(ii) yields that  $\dim_H \mathcal{V}_q < 1$ . This implies in particular that the set  $\mathcal{V}_q$  is a one-dimensional null set. Applying Theorem 1.2 (and the remark following its proof) and Fubini's theorem we conclude that  $\overline{\mathbf{U}}$  is a two-dimensional null set.

(i) Since  $\mathcal{U}_q$  is not closed for all  $q > 1$ ,  $\mathbf{U}$  cannot be closed. Since  $\overline{\mathbf{U}}$  is a two-dimensional null set, it has no interior points. It remains to show that  $\mathbf{U}$  (and thus  $\overline{\mathbf{U}}$ ) has no isolated points. If  $q > 1$  is an integer, then, as is well known,  $\mathcal{U}_q$  is dense

---

<sup>2</sup>We call a set  $X \subset \mathbb{R}$  *closed from above* if  $x$  belongs to  $X$  whenever there exists a sequence  $(x_n)$  of elements of  $X$  that converges to  $x$  from above.

in  $J_q = [0, 1]$ . If  $q > 1$  is a non-integer, then each  $(x, q) \in \mathbf{U}$  is not isolated because  $\mathcal{U}'_q \subset \mathcal{U}'_r$  whenever  $q < r$  and  $\lceil q \rceil = \lceil r \rceil$ .

(iii) From Corollary 7.10 in [9] we may conclude that for almost all  $q > 1$

$$\dim_H \mathcal{U}_q \leq \max \{0, \dim_H \mathbf{U} - 1\}$$

which would contradict Proposition 4.1(i) if we had  $\dim_H \mathbf{U} < 2$ .  $\square$

*Acknowledgements.* The first author has been supported by NWO, Project nr. ISK04G. Part of this work was done during a visit of the second author at the Department of Mathematics of the Delft University of Technology. He is grateful for this invitation and for the excellent working conditions.

#### REFERENCES

- [1] C. Baiocchi, V. Komornik, *Greedy and quasi-greedy expansions in non-integer bases*, arXiv:0710.3001 [math.], October 16, 2007.
- [2] K. Dajani, M. de Vries, *Invariant densities for random  $\beta$ -expansions*, J. Eur. Math. Soc. **9** (2007), no. 1, 157–176.
- [3] Z. Daróczy, I. Kátai, *Univoque sequences*, Publ. Math. Debrecen **42** (1993), no. 3–4, 397–407.
- [4] Z. Daróczy, I. Kátai, *On the structure of univoque numbers*, Publ. Math. Debrecen **46** (1995), no. 3–4, 385–408.
- [5] M. de Vries, V. Komornik, *Unique expansions of real numbers*, Adv. Math. **221** (2009), no. 2, 390–427.
- [6] P. Erdős, Horváth, M., I. Joó, *On the uniqueness of the expansions  $1 = \sum q^{-n_i}$* , Acta Math. Hungar. **58** (1991), no. 3–4, 333–342.
- [7] P. Erdős, I. Joó, V. Komornik, *Characterization of the unique expansions  $1 = \sum_{i=1}^{\infty} q^{-n_i}$  and related problems*, Bull. Soc. Math. France **118** (1990), no. 3, 377–390.
- [8] P. Erdős, I. Joó, V. Komornik, *On the number of  $q$ -expansions*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **37** (1994), 109–118.
- [9] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley & Sons, Chichester, second edition, 2003.
- [10] P. Glendinning, N. Sidorov, *Unique representations of real numbers in non-integer bases*, Math. Res. Lett. **8** (2001), no. 4, 535–543.
- [11] G. Kallós, *The structure of the univoque set in the small case*, Publ. Math. Debrecen **54** (1999), no. 1–2, 153–164.
- [12] G. Kallós, *The structure of the univoque set in the big case*, Publ. Math. Debrecen **59** (2001), no. 3–4, 471–489.
- [13] I. Kátai, G. Kallós, *On the set for which 1 is univoque*, Publ. Math. Debrecen **58** (2001), no. 4, 743–750.
- [14] V. Komornik, P. Loreti, *Unique developments in non-integer bases*, Amer. Math. Monthly **105** (1998), no. 7, 636–639.
- [15] V. Komornik, P. Loreti, *On the topological structure of univoque sets*, J. Number Theory, **122** (2007), no. 1, 157–183.
- [16] W. Parry, *On the  $\beta$ -expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.
- [17] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.
- [18] N. Sidorov, *Almost every number has a continuum of  $\beta$ -expansions*, Amer. Math. Monthly **110** (2003), no. 9, 838–842.
- [19] N. Sidorov, *Combinatorics of linear iterated function systems with overlaps*, Nonlinearity **20** (2007), 1299–1312.

DELFT UNIVERSITY OF TECHNOLOGY, MEKELWEG 4, 2628 CD DELFT, THE NETHERLANDS

E-mail address: w.m.devries@tudelft.nl

DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE

E-mail address: komornik@math.u-strasbg.fr

# A TWO-DIMENSIONAL UNIVOQUE SET

MARTIJN DE VRIES AND VILMOS KOMORNIK

**ABSTRACT.** Let  $\mathbf{J} \subset \mathbb{R}^2$  be the set of couples  $(x, q)$  with  $q > 1$  such that  $x$  has at least one representation of the form  $x = \sum_{i=1}^{\infty} c_i q^{-i}$  with integer coefficients  $c_i$  satisfying  $0 \leq c_i < q$ ,  $i \geq 1$ . In this case we say that  $(c_i) = c_1 c_2 \dots$  is an expansion of  $x$  in base  $q$ . Let  $\mathbf{U}$  be the set of couples  $(x, q) \in \mathbf{J}$  such that  $x$  has exactly one expansion in base  $q$ .

In this paper we deduce some topological and combinatorial properties of the set  $\mathbf{U}$ . We characterize the closure of  $\mathbf{U}$ , and we determine its Hausdorff dimension. For  $(x, q) \in \mathbf{J}$ , we also prove new properties of the lexicographically largest expansion of  $x$  in base  $q$ .

## 1. INTRODUCTION

Let  $\mathbf{J}$  be the set consisting of all elements  $(x, q) \in \mathbb{R} \times (1, \infty)$  such that there exists at least one sequence  $(c_i) = c_1 c_2 \dots$  of integers satisfying  $0 \leq c_i < q$  for all  $i$ , and

$$(1.1) \quad x = \frac{c_1}{q} + \frac{c_2}{q^2} + \dots.$$

If (1.1) holds, we say that  $(c_i)$  is an *expansion of  $x$  in base  $q$* , and if the base  $q$  is understood from the context, we sometimes simply say that  $(c_i)$  is an expansion of  $x$ . The numbers  $c_i$  of an expansion  $(c_i)$  are usually referred to as *digits*. We denote by  $\lceil q \rceil$  the smallest integer larger than or equal to  $q$ . The *alphabet*  $A_q$  is the set of “admissible” digits in base  $q$ , i.e.,  $A_q = \{0, \dots, \lceil q \rceil - 1\}$ .

If  $q > 1$  and  $0 \leq x \leq (\lceil q \rceil - 1)/(q - 1)$ , then a particular expansion of  $x$  in base  $q$ , the so-called *quasi-greedy expansion*  $(a_i(x, q))$ , may be defined recursively as follows. For  $x = 0$  we set  $(a_i(x, q)) := 0^\infty$ . If  $x > 0$  and  $a_i(x, q)$  has already been defined for  $1 \leq i < n$  (no condition if  $n = 1$ ), then  $a_n(x, q)$  is the largest element of  $A_q$  satisfying

$$\frac{a_1(x, q)}{q} + \dots + \frac{a_n(x, q)}{q^n} < x.$$

One easily verifies that  $(a_i(x, q))$  is indeed an expansion of  $x$  in base  $q$ . Therefore

$$(x, q) \in \mathbf{J} \iff q > 1 \quad \text{and} \quad x \in J_q := \left[0, \frac{\lceil q \rceil - 1}{q - 1}\right].$$

Let us denote by  $\mathbf{U}$  the set of couples  $(x, q) \in \mathbf{J}$  such that  $x$  has *exactly one* expansion in base  $q$ . For example,  $(0, q) \in \mathbf{U}$  for every  $q > 1$ , but  $\mathbf{U}$  has many more elements. The main purpose of this paper is to describe the topological and combinatorial nature of  $\mathbf{U}$ . We will prove the following theorem:

### Theorem 1.1.

- (i) *The set  $\mathbf{U}$  is not closed. Its closure  $\overline{\mathbf{U}}$  is a Cantor set*<sup>1</sup>.

*Date:* December 17, 2010.

*2000 Mathematics Subject Classification.* Primary:11A63, Secondary:11B83.

*Key words and phrases.* Greedy expansion, beta-expansion, univoque sequence, univoque set, Cantor set, Hausdorff dimension.

<sup>1</sup>We recall that a Cantor set is a nonempty closed set having neither interior nor isolated points.

- (ii) Both  $\mathbf{U}$  and  $\overline{\mathbf{U}}$  are two-dimensional Lebesgue null sets.
- (iii) Both  $\mathbf{U}$  and  $\overline{\mathbf{U}}$  have Hausdorff dimension two.

As far as we know this two-dimensional *univoque* set has not yet been investigated. There exists, however, a number of papers devoted to the study of its one-dimensional sections

$$\mathcal{U} := \{q > 1 : (1, q) \in \mathbf{U}\}$$

and

$$\mathcal{U}_q := \{x \in J_q : (x, q) \in \mathbf{U}\}, \quad q > 1.$$

The study of  $\mathcal{U}$  started with the paper of Erdős, Horváth and Joó [6] and was studied subsequently in [4], [5], [7], [8], [15], [16], [17]. We recall in particular that  $\mathcal{U}$  and its closure  $\overline{\mathcal{U}}$  have Lebesgue measure zero and Hausdorff dimension one.

The sets  $\mathcal{U}_q$  have been investigated in [3], [4], [5], [11], [13], [14]. It is known (see [5]) that  $\mathcal{U}_q$  is closed if and only if  $q$  does not belong to the null set  $\overline{\mathcal{U}}$ , and that its closure  $\overline{\mathcal{U}_q}$  has Lebesgue measure zero for all non-integer bases  $q > 1$ . Moreover, the set of numbers  $x \in J_q$  having a continuum of expansions in base  $q$  has full Lebesgue measure for each non-integer  $q > 1$  (see [2], [20], [21]).

The key to the proof of Theorem 1.1 is an algebraic characterization of  $\overline{\mathbf{U}}$  by using the quasi-greedy expansions  $(a_i(x, q))$ . We write for brevity  $\alpha_i(q) := a_i(1, q)$ ,  $i \in \mathbb{N} := \{1, 2, \dots\}$ ,  $q > 1$ . Note that  $\alpha_1(q) = \lceil q \rceil - 1$ , the largest admissible digit in base  $q$ . In the statement of the following theorem we use the lexicographic order between sequences and we define the *conjugate* (in base  $q$ ) of a digit  $c \in A_q$  by  $\bar{c} := \alpha_1(q) - c$ . If  $c_i \in A_q$ ,  $i \geq 1$ , we shall also write  $\overline{c_1 \dots c_n}$  instead of  $\bar{c}_1 \dots \bar{c}_n$  and  $\overline{c_1 c_2 \dots}$  instead of  $\bar{c}_1 \bar{c}_2 \dots$ .

**Theorem 1.2.** *A point  $(x, q) \in \mathbf{J}$  belongs to  $\overline{\mathbf{U}}$  if and only if*

$$\overline{a_{n+1}(x, q) a_{n+2}(x, q) \dots} \leq \alpha_1(q) \alpha_2(q) \dots \quad \text{whenever } a_n(x, q) > 0.$$

Along with the quasi-greedy expansion, we also need the notion of the *greedy expansion*  $(b_i(x, q))$  for  $x \in J_q$ , introduced by Rényi [19]. It can be defined by a slight modification of the above recursion: if  $b_i(x, q)$  has already been defined for all  $1 \leq i < n$  (no condition if  $n = 1$ ), then  $b_n(x, q)$  is the largest element of  $A_q$  satisfying

$$\frac{b_1(x, q)}{q} + \dots + \frac{b_n(x, q)}{q^n} \leq x.$$

Note that the greedy expansion  $(b_i(x, q))$  of a number  $x \in J_q$  is the lexicographically largest expansion of  $x$  in base  $q$ . We denote the greedy expansion of 1 in base  $q$  by  $(\beta_i(q)) := (b_i(1, q))$ .

The rest of this paper is organized as follows. In the next section we give a short overview of some basic results on greedy and quasi-greedy expansions, and we prove some new results concerning the coordinate-wise convergence of sequences of these expansions. We shall prove (see Theorem 2.7) that the set of numbers  $x \in J_q$  for which the greedy expansion of  $x$  in base  $q$  is not the greedy expansion of a number belonging to  $J_p$  in any smaller base  $p \in (1, q)$  is of full Lebesgue measure and its complement in  $J_q$  is a set of first category and Hausdorff dimension one. We shall also prove (see Theorem 2.8) that for each word  $v := b_{\ell+1}(x, q) \dots b_{\ell+m}(x, q)$  ( $\ell \geq 0, m \geq 1, x \in [0, 1)$ ) there exists a set  $Y_v \subset J_q$  of first category and Hausdorff dimension less than one, such that the word  $v$  occurs in the greedy expansion in base  $q$  of every number belonging to  $J_q \setminus Y_v$ . Using (some of) the results of Section 2 we prove Theorem 1.2 in Section 3 and Theorem 1.1 in Section 4.

## 2. GREEDY AND QUASI-GREEDY EXPANSIONS

In this paper we consider only one-sided sequences of nonnegative integers. We equip this set of sequences  $\{0, 1, \dots\}^{\mathbb{N}}$  with the topology of coordinate-wise convergence. We say that an expansion is *infinite* if it has infinitely many nonzero elements; otherwise it is called *finite*. Using this terminology, the quasi-greedy expansion  $(a_i(x, q))$  of a number  $x \in J_q \setminus \{0\}$  is the lexicographically largest *infinite* expansion of  $x$  in base  $q$ . Moreover, if the greedy expansion of  $x \in J_q$  is infinite, then  $(a_i(x, q)) = (b_i(x, q))$ .

The family of all quasi-greedy expansions is characterized by the following propositions (see [1] or [5] for a proof):

**Proposition 2.1.** *The map  $q \mapsto (\alpha_i(q))$  is an increasing bijection from the open interval  $(1, \infty)$  onto the set of all infinite sequences  $(\alpha_i)$  satisfying*

$$\alpha_{k+1}\alpha_{k+2}\dots \leq \alpha_1\alpha_2\dots \quad \text{for all } k \geq 1.$$

**Proposition 2.2.** *For each  $q > 1$ , the map  $x \mapsto (a_i(x, q))$  is an increasing bijection from  $J_q \setminus \{0\}$  onto the set of all infinite sequences  $(a_i)$  satisfying*

$$a_n \in A_q \quad \text{for all } n \geq 1$$

and

$$a_{n+1}a_{n+2}\dots \leq \alpha_1(q)\alpha_2(q)\dots \quad \text{whenever } a_n < \alpha_1(q).$$

The quasi-greedy expansions have a lower semicontinuity property for the order topology induced by the lexicographic order. More precisely, we have the following result.

**Lemma 2.3.** *Let  $(x, q) \in \mathbf{J}$  and  $(y_n, r_n) \in \mathbf{J}$ ,  $n \in \mathbb{N}$ . Then*

- (i) *for each positive integer  $m$  there exists a neighborhood  $\mathbf{W} \subset \mathbb{R}^2$  of  $(x, q)$  such that*

$$(2.1) \quad a_1(y, r) \dots a_m(y, r) \geq a_1(x, q) \dots a_m(x, q) \quad \text{for all } (y, r) \in \mathbf{W} \cap \mathbf{J};$$

- (ii) *if  $y_n \uparrow x$  and  $r_n \uparrow q$ , then  $(a_i(y_n, r_n))$  converges to  $(a_i(x, q))$ .*

*Proof.* (i) We may assume that  $x \neq 0$ . By definition of the quasi-greedy expansion we have

$$\sum_{i=1}^n \frac{a_i(x, q)}{q^i} < x \quad \text{for all } n = 1, 2, \dots$$

For any fixed positive integer  $m$ , if  $(y, r) \in \mathbf{J}$  is sufficiently close to  $(x, q)$ , then  $r > [q] - 1$ , i.e.,  $A_q \subset A_r$ , and

$$\sum_{i=1}^n \frac{a_i(x, q)}{r^i} < y, \quad n = 1, \dots, m.$$

These inequalities imply (2.1).

(ii) If  $y_n \leq x$  and  $r_n \leq q$ , we deduce from the definition of the quasi-greedy expansion that

$$(a_i(x, q)) \geq (a_i(y_n, r_n))$$

for every  $n$ . Equivalently, we have

$$a_1(x, q) \dots a_m(x, q) \geq a_1(y_n, r_n) \dots a_m(y_n, r_n)$$

for all positive integers  $m$  and  $n$ . It remains to notice that by the previous part the converse inequality also holds for each fixed  $m$  if  $n$  is large enough.  $\square$

The family of greedy expansions has already been characterized by Parry [18]:

**Proposition 2.4.** *For a given base  $q > 1$ , the map  $x \mapsto (b_i(x, q))$  is an increasing bijection from  $J_q$  onto the set of all sequences  $(b_i)$  satisfying*

$$b_n \in A_q \quad \text{for all } n \geq 1$$

and

$$b_{n+1}b_{n+2}\dots < \alpha_1(q)\alpha_2(q)\dots \quad \text{whenever } b_n < \alpha_1(q).$$

The greedy expansions have the following upper semicontinuity property:

**Lemma 2.5.** *Let  $(x, q) \in \mathbf{J}$ ,  $(y_n, r_n) \in \mathbf{J}$ ,  $n \in \mathbb{N}$  and suppose  $q \notin \mathbb{N}$ . Then*

(i) *for each positive integer  $m$  there exists a neighborhood  $\mathbf{W} \subset \mathbb{R}^2$  of  $(x, q)$  such that*

$$(2.2) \quad b_1(y, r) \dots b_m(y, r) \leq b_1(x, q) \dots b_m(x, q) \quad \text{for all } (y, r) \in \mathbf{W} \cap \mathbf{J};$$

(ii) *if  $y_n \downarrow x$  and  $r_n \downarrow q$ , then  $(b_i(y_n, r_n))$  converges to  $(b_i(x, q))$ .*

*Proof.* (i) By the definition of greedy expansions we have

$$\sum_{i=1}^n \frac{b_i(x, q)}{q^i} > x - \frac{1}{q^n} \quad \text{whenever } b_n(x, q) < \alpha_1(q).$$

If  $(y, r) \in \mathbf{J}$  is sufficiently close to  $(x, q)$ , then  $A_r = A_q$ ,  $\alpha_1(r) = \alpha_1(q)$ , and

$$\sum_{i=1}^n \frac{b_i(x, q)}{r^i} > y - \frac{1}{r^n} \quad \text{whenever } n \leq m \text{ and } b_n(x, q) < \alpha_1(r).$$

These inequalities imply (2.2).

(ii) If  $y_n \geq x$  and  $r_n \geq q$ , we deduce from the definition of the greedy expansion that

$$(b_i(x, q)) \leq (b_i(y_n, r_n))$$

for every  $n$ . Equivalently, we have

$$b_1(x, q) \dots b_m(x, q) \leq b_1(y_n, r_n) \dots b_m(y_n, r_n)$$

for all positive integers  $m$  and  $n$ . It remains to notice that by the previous part the converse inequality also holds for each fixed  $m$  if  $n$  is large enough.  $\square$

From Lemmas 2.3 and 2.5 we deduce the following result:

**Proposition 2.6.** *Consider  $(x, q) \in \mathbf{J}$  with a non-integer base  $q$  and assume that the greedy expansion  $(b_i(x, q))$  is infinite. If  $(y_n, r_n)$  converges to  $(x, q)$  in  $\mathbf{J}$ , then both  $(a_i(y_n, r_n))$  and  $(b_i(y_n, r_n))$  converge to  $(b_i(x, q)) = (a_i(x, q))$ .*

*Proof.* For each positive integer  $m$  there exists a neighborhood  $\mathbf{W} \subset \mathbb{R}^2$  of  $(x, q)$  such that for all  $(y, r) \in \mathbf{W} \cap \mathbf{J}$ ,

$$\begin{aligned} a_1(x, q) \dots a_m(x, q) &\leq a_1(y, r) \dots a_m(y, r) \\ &\leq b_1(y, r) \dots b_m(y, r) \\ &\leq b_1(x, q) \dots b_m(x, q). \end{aligned}$$

The result follows from our assumption that  $(a_i(x, q)) = (b_i(x, q))$ .  $\square$

**Theorem 2.7.** *Let  $q > 1$  be a real number. Then*

(i) *for each  $r \in (1, q)$ , the Hausdorff dimension of the set*

$$G_{r, q} := \left\{ \sum_{i=1}^{\infty} \frac{b_i(x, r)}{q^i} : x \in J_r \right\}$$

*equals  $\log r / \log q$ ;*

(ii) *the set*

$$G_q := \bigcup \{G_{r,q} : r \in (1, q)\}$$

*is of first category, has Lebesgue measure zero and Hausdorff dimension one.*

*Proof.* (i) It is well known (see, e.g., [17], [18]) and easy to prove that the set of numbers  $r > 1$  for which  $(\beta_i(r))$  is finite is dense in  $[1, \infty)$ . Moreover, if  $(\beta_i(r))$  is finite and  $\beta_n(r)$  is its last nonzero element, then  $(\alpha_i(r)) = (\beta_1(r) \dots \beta_{n-1}(r) \beta_n^-(r))^\infty$  ( $\beta_n^-(r) := \beta_n(r) - 1$ ). By virtue of Propositions 2.1 and 2.4 we have  $G_{s,q} \subset G_{t,q}$  whenever  $1 < s < t < q$ . Hence it is enough to prove that  $\dim_H G_{r,q} = \log r / \log q$  for those values  $r \in (1, q)$  for which  $(\alpha_i(r))$  is periodic.

Fix  $r \in (1, q)$  such that  $(\alpha_i) := (\alpha_i(r))$  is periodic and let  $n \in \mathbb{N}$  be such that  $(\alpha_i) = (\alpha_1 \dots \alpha_n)^\infty$ . Let us denote by  $W_r$  the set consisting of the finite words

$$w_{ij} := \alpha_1 \dots \alpha_{j-1} i, \quad 0 \leq i < \alpha_j, \quad 1 \leq j \leq n$$

and

$$w_{\alpha_n n} := \alpha_1 \dots \alpha_{n-1} \alpha_n.$$

Let  $\mathcal{F}'_r$  be the set of sequences  $(c_i) = c_1 c_2 \dots$  such that for each  $k \geq 0$  the inequality  $c_{k+1} \dots c_{k+n} \leq \alpha_1 \dots \alpha_n$  holds. Note that the set  $\mathcal{F}'_r$  consists of those sequences  $(c_i)$  such that each tail of  $(c_i)$  (including  $(c_i)$  itself) starts with a word belonging to  $W_r$ . It follows from Propositions 2.1 and 2.4 that a sequence  $(b_i)$  is greedy in base  $r$  if and only if  $b_m \in A_r$  for all  $m \geq 1$  and

$$b_{m+k+1} b_{m+k+2} \dots < \alpha_1 \alpha_2 \dots \quad \text{for all } k \geq 0, \text{ whenever } b_m < \alpha_1.$$

Therefore, any greedy expansion  $(b_i) \neq \alpha_1^\infty$  in base  $r$  can be written as  $\alpha_1^\ell c_1 c_2 \dots$  for some  $\ell \geq 0$  ( $\alpha_1^0$  denotes the empty word) and some sequence  $(c_i)$  belonging to  $\mathcal{F}'_r$ . Conversely, if no tail of a sequence belonging to  $\mathcal{F}'_r$  equals  $(\alpha_i)$ , then it is the greedy expansion in base  $r$  of some  $x \in J_r$ . Hence if we set

$$\mathcal{F}_{r,q} := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{q^i} : (c_i) \in \mathcal{F}'_r \right\},$$

then  $\mathcal{F}_{r,q} \setminus G_{r,q}$  is countable and  $G_{r,q}$  can be covered by countably many sets similar to  $\mathcal{F}_{r,q}$ . Since the union of countably many sets of Hausdorff dimension  $s$  is still of Hausdorff dimension  $s$ , we have  $\dim_H G_{r,q} = \dim_H \mathcal{F}_{r,q}$ .

We associate with each word  $w_{ij} \in W_r$  a similarity  $S_{ij} : J_q \rightarrow J_q$  defined by the formula

$$S_{ij}(x) := \frac{\alpha_1}{q} + \dots + \frac{\alpha_{j-1}}{q^{j-1}} + \frac{i}{q^j} + \frac{x}{q^j}, \quad x \in J_q.$$

It follows from Proposition 2.1 and the definition of  $\mathcal{F}_{r,q}$  that

$$(2.3) \quad \mathcal{F}_{r,q} = \bigcup S_{ij}(\mathcal{F}_{r,q})$$

where the union runs over all  $i$  and  $j$  for which  $w_{ij} \in W_r$ . Applying Proposition 2.1 again, it follows that  $r$  is the largest element of the set of numbers  $t > 1$  for which  $\alpha_i(t) = \alpha_i$ ,  $1 \leq i \leq n$ . Hence  $\alpha_1 \dots \alpha_n < \alpha_1(q) \dots \alpha_n(q)$  and therefore each sequence in  $\mathcal{F}'_r$  is the greedy expansion in base  $q$  of some  $x \in \mathcal{F}_{r,q}$ . It follows that the sets  $S_{ij}(\mathcal{F}_{r,q})$  on the right side of (2.3) are disjoint. Moreover, the function  $x \mapsto (b_i(x, q))$  that maps  $\mathcal{F}_{r,q}$  onto  $\mathcal{F}'_r$  is increasing. Using the definition of  $\mathcal{F}'_r$  it is easily seen that the limit of each monotonic sequence of elements in  $\mathcal{F}_{r,q}$  belongs to  $\mathcal{F}_{r,q}$ . We conclude that the closed set  $\mathcal{F}_{r,q}$  is the (nonempty compact) invariant set of this system of similarities. An application of Propositions 9.6 and 9.7 in [9] yields that

$$\dim_H \mathcal{F}_{r,q} = \dim_H G_{r,q} = s$$



where  $s$  is the real solution of the equation

$$\frac{\alpha_1}{q^s} + \dots + \frac{\alpha_{n-1}}{q^{(n-1)s}} + \frac{\alpha_n + 1}{q^{ns}} = 1.$$

Since

$$\frac{\alpha_1}{r} + \dots + \frac{\alpha_{n-1}}{r^{n-1}} + \frac{\alpha_n + 1}{r^n} = 1$$

we have  $s = \log r / \log q$ .

(ii) It follows at once from Theorem 2.7(i) that  $\dim_H G_q = 1$ . Let  $r \in (1, q)$  be such that  $(\alpha_i(r))$  is periodic. The proof of Theorem 2.7(i) shows that

$$G_{r,q} \subset \bigcup_{n=1}^{\infty} (a_n + b_n \mathcal{F}_{r,q})$$

for some constants  $a_n, b_n \in \mathbb{R}$  ( $n \in \mathbb{N}$ ). Since  $\mathcal{F}_{r,q}$  is a closed set of Hausdorff dimension less than one, it follows in particular that the sets  $a_n + b_n \mathcal{F}_{r,q}$  are nowhere dense null sets. Since  $G_{s,q} \subset G_{t,q}$  whenever  $1 < s < t < q$ , the set  $G_q$  is a null set of first category.  $\square$

**Theorem 2.8.** *Let  $q > 1$  be a real number.*

- (i) *Let  $v := b_{\ell+1}(y, q) \dots b_{\ell+m}(y, q)$  for some  $y \in [0, 1)$  and some integers  $\ell \geq 0$  and  $m \geq 1$ . The set  $Y_v$  of numbers  $x \in J_q$  for which the word  $v$  does not occur in the greedy expansion of  $x$  in base  $q$  has Hausdorff dimension less than one.*
- (ii) *The set  $Y$  of numbers  $x \in J_q$  for which at least one word of the form  $b_{\ell+1}(y, q) \dots b_{\ell+m}(y, q)$  ( $\ell \geq 0, m \geq 1, y \in [0, 1)$ ) does not occur in the greedy expansion of  $x$  in base  $q$  is of first category, has Lebesgue measure zero and Hausdorff dimension one.*

*Proof.* (i) Using the inequality  $(b_i(y, q)) < (\alpha_i(q))$ , it follows from Proposition 2.4 that for some  $k \in \mathbb{N}$ , there exist positive integers  $m_1, \dots, m_k$  and nonnegative integers  $\ell_1, \dots, \ell_k$  satisfying  $\alpha_{m_j}(q) > 0$  and  $\ell_j < \alpha_{m_j}(q)$  for each  $1 \leq j \leq k$ , such that  $v$  is a subword of

$$w := \alpha_1(q) \dots \alpha_{m_1-1}(q) \ell_1 \dots \alpha_1(q) \dots \alpha_{m_k-1}(q) \ell_k.$$

Let  $W_q$  and  $\mathcal{F}'_q$  be the same as the sets  $W_r$  and  $\mathcal{F}'_r$  defined in the proof of the previous theorem, but now with  $(\alpha_i) := (\alpha_i(q))$  and  $n \geq \max\{m_1, \dots, m_k\}$  large enough such that the inequality

$$(2.4) \quad \left(1 + \frac{1}{q^n}\right)^k < 1 + \frac{1}{q^{m_1+\dots+m_k}}$$

holds. If  $w_{i_1 j_1}, \dots, w_{i_k j_k}$  are  $k$  words belonging to  $W_q$  such that

$$i_1 j_1 \dots i_k j_k \neq \ell_1 m_1 \dots \ell_k m_k,$$

we associate with them a similarity  $S_{i_1 j_1 \dots i_k j_k} : J_q \rightarrow J_q$  defined by the formula

$$\begin{aligned} S_{i_1 j_1 \dots i_k j_k}(x) &= \frac{\alpha_1}{q} + \dots + \frac{\alpha_{j_1-1}}{q^{j_1-1}} + \frac{i_1}{q^{j_1}} \\ &\quad + \frac{\alpha_1}{q^{j_1+1}} + \dots + \frac{\alpha_{j_2-1}}{q^{j_1+j_2-1}} + \frac{i_2}{q^{j_1+j_2}} \\ &\quad \vdots \\ &\quad + \frac{\alpha_1}{q^{j_1+\dots+j_{k-1}+1}} + \dots + \frac{\alpha_{j_k-1}}{q^{j_1+\dots+j_k-1}} + \frac{i_k}{q^{j_1+\dots+j_k}} \\ &\quad + \frac{x}{q^{j_1+\dots+j_k}}, \quad x \in J_q. \end{aligned}$$

Let  $\mathcal{G}'_q$  denote the set of those sequences belonging to  $\mathcal{F}'_q$  which do not contain the word  $w$ , and let

$$\mathcal{G}_q := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{q^i} : (c_i) \in \mathcal{G}'_q \right\}.$$

Since  $(\alpha_i) = (\alpha_i(q))$ , a sequence belonging to  $\mathcal{F}'_q$  is not necessarily the greedy expansion in base  $q$  of a number  $x \in J_q$ , but this does not affect our proof. It is important, however, that any greedy expansion  $(b_i) \neq \alpha_1^\infty$  in base  $q$  can be written as  $\alpha_1^\ell c_1 c_2 \dots$  for some  $\ell \geq 0$  and some sequence  $(c_i)$  belonging to  $\mathcal{F}'_q$ . If  $Y_w$  denotes the set of numbers  $x \in J_q$  for which the word  $w$  does not occur in  $(b_i(x, q))$  then the latter fact implies that the set  $Y_w \setminus \{\alpha_1/(q-1)\}$  can be covered by countably many sets similar to  $\mathcal{G}_q$ .

It follows from the definition of  $\mathcal{G}_q$  that

$$\mathcal{G}_q \subset \bigcup S_{i_1 j_1 \dots i_k j_k}(\mathcal{G}_q)$$

where the union runs over all  $i_1 j_1 \dots i_k j_k$  for which the similarity  $S_{i_1 j_1 \dots i_k j_k}$  is defined above. Hence

$$\overline{\mathcal{G}_q} \subset \bigcup S_{i_1 j_1 \dots i_k j_k}(\overline{\mathcal{G}_q})$$

and thus  $\mathcal{G}_q \subset \mathcal{H}_q$  where  $\mathcal{H}_q$  is the (nonempty compact) invariant set of this system of similarities. Let  $\tilde{\alpha}_i := \alpha_i$  for  $1 \leq i < n$  and  $\tilde{\alpha}_n := \alpha_n + 1$ . From Proposition 9.6 in [9] we know that  $\dim_H \mathcal{H}_q \leq s$  where  $s$  is the real solution of the equation

$$(2.5) \quad \sum_{j_1=1}^n \sum_{j_2=1}^n \dots \sum_{j_k=1}^n \left( \frac{\prod_{i=1}^k \tilde{\alpha}_{j_i}}{q^{(j_1+\dots+j_k)s}} \right) - \frac{1}{q^{(m_1+\dots+m_k)s}} = 1.$$

Denoting the left side of (2.5) by  $C(s)$ , we have

$$C(1) + \frac{1}{q^{m_1+\dots+m_k}} = \left( \sum_{i=1}^n \frac{\tilde{\alpha}_i}{q^i} \right)^k < \left( 1 + \frac{1}{q^n} \right)^k.$$

By (2.4) we have  $C(1) < 1$ , and thus  $\dim_H Y_v \leq \dim_H Y_w \leq \dim_H \mathcal{H}_q < 1$ .

(ii) The proof of Theorem 2.8(i) shows that

$$Y_v \subset Y_w \subset \bigcup_{n=1}^{\infty} (c_n + d_n \mathcal{H}_q)$$

for some constants  $c_n, d_n \in \mathbb{R}$  ( $n \in \mathbb{N}$ ). Arguing as in the proof of Theorem 2.7(ii) we may conclude that  $Y_v$  is a null set of first category. Since  $Y$  is a countable union of sets of the form  $Y_v$  the same properties hold for the set  $Y$ . Let  $r \in (1, q)$  and let  $G_{r,q}$  be the set defined in Theorem 2.7. Due to Theorem 2.7(i) it is now sufficient to show that  $G_{r,q} \subset Y$ . By Proposition 2.1 there exists an integer  $n \in \mathbb{N}$  such that the inequality  $\alpha_1(r) \dots \alpha_n(r) < \alpha_1(q) \dots \alpha_n(q)$ . Note that the greedy expansion in base  $q$  of a number  $x \in G_{r,q}$  equals  $(b_i(x', r))$  for some  $x' \in J_r$  by Proposition 2.4. Applying Propositions 2.1 and 2.4 once more we conclude that the sequence  $0\alpha_1(q) \dots \alpha_n(q)0^\infty$  equals  $(b_i(y, q))$  for some  $y \in [0, 1)$  while the word  $0\alpha_1(q) \dots \alpha_n(q)$  does not occur in the greedy expansion in base  $r$  of any number belonging to  $J_r$ .  $\square$

*Remark.* In this remark we will briefly sketch a proof of Theorem 2.7(i) and Theorem 2.8(i) that was pointed out to us by the anonymous referee. For  $q > 1$ , let  $B_n(q)$  be the number of possible blocks of length  $n$  that may occur in  $(b_i(x, q))$  for some  $x \in J_q$ . Since  $b_{n+1}(x, q)b_{n+2}(x, q) \dots$  is the greedy expansion of  $\sum_{i=1}^{\infty} b_{n+i}q^{-i}$  for each  $n \in \mathbb{N}$  and  $x \in J_q$ , we have

$$B_n(q) = |\{(b_1(x, q), \dots, b_n(x, q)) : x \in J_q\}|.$$

Let the  $q$ -shift  $\sigma_q$  be the one-sided left shift on the set  $\{(b_i(x, q)) : x \in J_q\}$ . It is well known (see [12]) that the topological entropy  $h_{\text{top}}(\sigma_q)$  of the  $q$ -shift, given by

$$(2.6) \quad h_{\text{top}}(\sigma_q) := \lim_{n \rightarrow \infty} \frac{\log(B_n(q))}{n},$$

equals  $\log q$ . By some modifications of the proof of Proposition III.1 in [10], one shows that  $\dim_H G_{r,q} = h_{\text{top}}(\sigma_r)/\log q = \log r/\log q$ . Theorem 2.8(i) may also be deduced from (2.6) and Proposition III.1 in [10]. On the other hand, our proof of these results enables us to show that the sets  $G_q$  and  $Y$  in Theorem 2.7(ii) and Theorem 2.8(ii) are of first category. Moreover, Theorem 2.7(i) combined with the formula  $\dim_H G_{r,q} = h_{\text{top}}(\sigma_r)/\log q$  gives an alternative proof of the fact that  $h_{\text{top}}(\sigma_q) = \log q$  for each  $q > 1$ .

### 3. PROOF OF THEOREM 1.2

The following characterization of unique expansions readily follows from Proposition 2.4.

**Proposition 3.1.** *Fix  $q > 1$ . A sequence  $(c_i)$  of integers  $c_i \in A_q$  is the unique expansion of some  $x \in J_q$  if and only if*

$$c_{n+1}c_{n+2}\dots < \alpha_1(q)\alpha_2(q)\dots \quad \text{whenever } c_n < \alpha_1(q)$$

and

$$\overline{c_{n+1}c_{n+2}\dots} < \alpha_1(q)\alpha_2(q)\dots \quad \text{whenever } c_n > 0.$$

In what follows we use the notation  $(a_i(x, q))$ ,  $(b_i(x, q))$ ,  $(\alpha_i(q))$  and  $(\beta_i(q))$  as introduced in Section 1. If  $x$  and  $q$  are clear from the context, then we omit these arguments and we simply write  $a_i$ ,  $b_i$ ,  $\alpha_i$  and  $\beta_i$ . If two couples  $(x, q)$  and  $(x', q')$  are considered simultaneously, then we also write  $a'_i$ ,  $b'_i$ ,  $\alpha'_i$  and  $\beta'_i$  in place of  $a_i(x', q')$ ,  $b_i(x', q')$ ,  $\alpha_i(q')$  and  $\beta_i(q')$ .

**Lemma 3.2.** *Given  $(x, q) \in \mathbf{J}$ , the following two conditions are equivalent:*

$$\begin{aligned} \overline{a_{n+1}a_{n+2}\dots} &\leq \alpha_1\alpha_2\dots \quad \text{whenever } a_n > 0; \\ \overline{a_{n+1}a_{n+2}\dots} &\leq \beta_1\beta_2\dots \quad \text{whenever } a_n > 0. \end{aligned}$$

*Proof.* Since  $(\alpha_i) \leq (\beta_i)$ , it suffices to show that if there exists a positive integer  $n$  such that

$$a_n > 0 \quad \text{and} \quad \overline{a_{n+1}a_{n+2}\dots} > \alpha_1\alpha_2\dots,$$

then there exists also a positive integer  $m$  such that

$$a_m > 0 \quad \text{and} \quad \overline{a_{m+1}a_{m+2}\dots} > \beta_1\beta_2\dots$$

If the greedy expansion  $(\beta_i)$  is infinite, then  $(\beta_i) = (\alpha_i)$  and we may choose  $m = n$ . If  $(\beta_i)$  has a last nonzero digit  $\beta_\ell$ , then  $(\alpha_i) = (\alpha_1\dots\alpha_\ell)^\infty$  with  $\alpha_1\dots\alpha_{\ell-1}\alpha_\ell = \beta_1\dots\beta_{\ell-1}\beta_\ell^-$  ( $\beta_\ell^- := \beta_\ell - 1$ ), and thus  $\alpha_\ell < \alpha_1$ . Since we have

$$\overline{a_{n+1}a_{n+2}\dots} > (\alpha_1\dots\alpha_\ell)^\infty$$

by assumption, there exists a nonnegative integer  $j$  satisfying

$$\overline{a_{n+1}\dots a_{n+j\ell}} = (\alpha_1\dots\alpha_\ell)^j \quad \text{and} \quad \overline{a_{n+j\ell+1}\dots a_{n+(j+1)\ell}} > \alpha_1\dots\alpha_\ell.$$

Putting  $m := n + j\ell$  it follows that

$$a_m > 0 \quad \text{and} \quad \overline{a_{m+1}\dots a_{m+\ell}} \geq \beta_1\dots\beta_\ell.$$

It follows from our assumption  $\overline{a_{n+1}a_{n+2}\dots} > \alpha_1\alpha_2\dots$  that  $(\alpha_i) < \alpha_1^\infty$  and  $(a_i) \neq \alpha_1^\infty$ . It follows from Proposition 2.2 that  $(a_i)$  has no tail equal to  $\alpha_1^\infty$ , so that  $\overline{a_{m+\ell+1}a_{m+\ell+2}\dots} > 0^\infty$ . We conclude that

$$\overline{a_{m+1}a_{m+2}\dots} > \beta_1\beta_2\dots \quad \square$$

*Definition.* We say that  $(x, q) \in \mathbf{J}$  belongs to the set  $\mathbf{V}$  if one of the two equivalent conditions of the preceding lemma is satisfied. Moreover, we define

$$\mathcal{V}_q := \{x \in J_q : (x, q) \in \mathbf{V}\}, \quad q > 1.$$

It follows from Proposition 3.1 that  $\mathbf{U} \subset \mathbf{V} \subset \mathbf{J}$ .

*Proof of Theorem 1.2.* We need to prove that  $\overline{\mathbf{U}} \cap \mathbf{J} = \mathbf{V}$ .

First we show that  $\mathbf{V} \subset \overline{\mathbf{U}}$ . To this end we introduce for each fixed  $q > 1$  the sets  $\mathcal{U}'_q$  and  $\mathcal{V}'_q$ , defined by

$$\mathcal{U}'_q := \{(a_i(x, q)) : x \in \mathcal{U}_q\} \quad \text{and} \quad \mathcal{V}'_q := \{(a_i(x, q)) : x \in \mathcal{V}_q\}.$$

Observe that  $\mathcal{U}'_q$  is simply the set of unique expansions in base  $q$ . It follows easily from Propositions 2.1, 2.2 and 3.1 that  $\mathcal{U}'_q \subset \mathcal{V}'_q$  for each  $q > 1$ , and that  $\mathcal{V}'_q \subset \mathcal{U}'_r$  for each  $r > q$  such that  $\lceil q \rceil = \lceil r \rceil$ . Since we also have  $\overline{\mathcal{U}_q} = \mathcal{V}_q = [0, 1]$  if  $q > 1$  is an integer, the result follows.

Next we show that  $\overline{\mathbf{U}} \cap \mathbf{J} \subset \mathbf{V}$ . Since  $\mathbf{U} \subset \mathbf{V}$  it is sufficient to prove that if  $(x, q) \in \mathbf{J} \setminus \mathbf{V}$ , then  $(x', q') \notin \mathbf{V}$  for all  $(x', q') \in \mathbf{J}$  close enough to  $(x, q)$ . By Lemma 3.2 there exist two positive integers  $n$  and  $m$  such that

$$(3.1) \quad a_n > 0 \quad \text{and} \quad \overline{a_{n+1} \dots a_{n+m}} > \beta_1 \dots \beta_m.$$

This implies in particular that  $q$  is not an integer, because otherwise  $(\alpha_i) = (\beta_i) = \beta_1^\infty$ . Hence, if  $q'$  is sufficiently close to  $q$ , then

$$(3.2) \quad \beta'_1 \dots \beta'_m \leq \beta_1 \dots \beta_m$$

by Lemma 2.5. It follows from the definition of quasi-greedy expansions that

$$\frac{a_1}{q} + \dots + \frac{a_{j-1}}{q^{j-1}} + \frac{a_j^+}{q^j} + \frac{1}{q^{j+m}} > x \quad \text{whenever } a_j < \alpha_1,$$

where  $a_j^+ := a_j + 1$ . If  $(x', q') \in \mathbf{J}$  is sufficiently close to  $(x, q)$ , then  $\alpha_1 = \alpha'_1$ , the inequality (3.2) is satisfied,  $a'_1 \dots a'_{n+m} \geq a_1 \dots a_{n+m}$  by Lemma 2.3, and

$$(3.3) \quad \frac{a_1}{q'} + \dots + \frac{a_{j-1}}{(q')^{j-1}} + \frac{a_j^+}{(q')^j} + \frac{1}{(q')^{j+m}} > x' \quad \text{whenever } j \leq n+m \text{ and } a_j < \alpha_1.$$

Now we distinguish between two cases.

If  $a'_1 \dots a'_{n+m} = a_1 \dots a_{n+m}$ , then we have

$$a'_n > 0 \quad \text{and} \quad \overline{a'_{n+1} \dots a'_{n+m}} > \beta_1 \dots \beta_m \geq \beta'_1 \dots \beta'_m$$

by (3.1) and (3.2). This proves that  $(x', q') \notin \mathbf{V}$ .

If  $a'_1 \dots a'_{n+m} > a_1 \dots a_{n+m}$ , then let us consider the smallest  $j$  for which  $a'_j > a_j$ . It follows from (3.2) and (3.3) that

$$a'_j = a_j^+ > 0 \quad \text{and} \quad \overline{a'_{j+1} \dots a'_{j+m}} = \beta_1^m > \beta_1 \dots \beta_m \geq \beta'_1 \dots \beta'_m.$$

Hence  $(x', q') \notin \mathbf{V}$  again.  $\square$

*Remark.* It is the purpose of this remark to describe the set  $\overline{\mathbf{U}} \setminus \mathbf{J}$ . For each  $m \in \mathbb{N}$ , we define the number  $q_m \in (m, m+1)$  by the equation

$$1 = \frac{m}{q_m} + \frac{1}{q_m^2}.$$

Fix  $q \in (m, q_m]$ . Since  $\alpha_1(q) = m$  and  $\alpha_2(q) = 0$ , Proposition 3.1 implies that a sequence  $(c_i) \in \{0, \dots, m\}^{\mathbb{N}}$  belongs to  $\mathcal{U}'_q$  if and only if for each  $n \in \mathbb{N}$ , we have

$$c_n < m \implies c_{n+1} < m$$

and

$$c_n > 0 \implies c_{n+1} > 0.$$

Denoting the set of all such sequences by  $D'_m$  and putting for  $m > 1$  (note that  $D'_1 = \{0^\infty, 1^\infty\}$ ),

$$D_m := \left\{ \sum_{i=1}^{\infty} \frac{c_i}{m^i} : (c_i) \in D'_m \right\},$$

one may verify that

$$\overline{\mathbf{U}} \setminus \mathbf{J} = \{(0, 1)\} \cup \bigcup_{m=2}^{\infty} (D_m \setminus [0, 1]) \times \{m\}.$$

For  $x \geq 0$ , let  $\mathcal{U}(x) = \{q > 1 : (x, q) \in \mathbf{U}\}$ , and denote its closure by  $\overline{\mathcal{U}(x)}$ . Using this notation, the set  $\mathcal{U}$  introduced in Section 1 equals  $\mathcal{U}(1)$ . The following corollary implies in particular that the sets  $\overline{\mathcal{U}(x)} \setminus \mathcal{U}(x)$  are (at most) countable.

**Corollary 3.3.** *Each element  $q \in \overline{\mathcal{U}(x)} \setminus \mathcal{U}(x)$  is algebraic over the field  $\mathbb{Q}(x)$ .*

*Proof.* If  $q \in \overline{\mathcal{U}(x)} \setminus \mathcal{U}(x)$  and  $q \notin \mathbb{N}$ , then  $(x, q) \in \mathbf{J}$  and thus  $(x, q) \in \mathbf{V}$  by Theorem 1.2. If the sequence  $(b_i(x, q))$  is infinite, then it ends with  $\alpha_1(q)\alpha_2(q)\dots$  as follows from the definition of  $\mathbf{V}$  and Propositions 2.4 and 3.1. Hence  $x$  has a finite expansion in base  $q$  or  $x$  can be written as

$$x = \frac{b_1(x, q)}{q} + \dots + \frac{b_n(x, q)}{q^n} + \frac{1}{q^n} \left( \frac{\alpha_1}{q-1} - 1 \right)$$

for some  $n \geq 0$ , whence  $q$  is algebraic over  $\mathbb{Q}(x)$ .  $\square$

#### 4. PROOF OF THEOREM 1.1

We need some results on the Hausdorff dimension of the sets  $\mathcal{U}_q$  and  $\mathcal{V}_q$  for  $q > 1$ . It follows from Theorem 1.2 that  $\mathcal{U}_q \subset \overline{\mathcal{U}_q} \subset \mathcal{V}_q$ . Moreover, if an element  $x \in \mathcal{V}_q \setminus \mathcal{U}_q$  has an infinite greedy expansion in base  $q$ , then  $(b_i(x, q))$  must end with  $\alpha_1(q)\alpha_2(q)\dots$  as follows from Propositions 2.4 and 3.1; hence  $\mathcal{V}_q \setminus \mathcal{U}_q$  is (at most) countable and the sets  $\mathcal{U}_q$ ,  $\overline{\mathcal{U}_q}$  and  $\mathcal{V}_q$  have the same Hausdorff dimension for each  $q > 1$ . Proposition 4.1 below is contained in the works of Daróczy and Kátai [4], Kallós [13], [14], Glendinning and Sidorov [11], and Sidorov [21]; for the reader's convenience we provide here an elementary proof.

**Proposition 4.1.** *We have*

- (i)  $\lim_{q \uparrow 2} \dim_H \mathcal{U}_q = 1$ ;
- (ii)  $\dim_H \mathcal{U}_q < 1$  for all non-integer  $q > 1$ .

*Proof.* (i) Assume that  $q \in (1, 2)$  is larger than the tribonacci number, i.e.,

$$\frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^3} < 1,$$

and let  $N = N(q) \geq 2$  be the largest integer satisfying

$$\frac{1}{q} + \dots + \frac{1}{q^{2N-1}} < 1.$$

Hence  $\alpha_1(q) = \dots = \alpha_{2N-1}(q) = 1$ . Let us denote by  $\mathcal{I}_q$  the set of numbers  $x \in J_q$  which have an expansion  $(c_i)$  in base  $q$  satisfying  $0 < c_{kN+1} + \dots + c_{(k+1)N} < N$  for every nonnegative integer  $k$ . Since in such expansions  $(c_i)$ , a zero (one) is followed by at most  $2N - 2$  consecutive one (zero) digits, it follows from Proposition 3.1 that  $\mathcal{I}_q \subset \mathcal{U}_q$ . It suffices now to prove that

$$(4.1) \quad \dim_H \mathcal{I}_q = \frac{\log(2^N - 2)}{N \log q};$$

indeed,  $q \uparrow 2$  implies  $N \rightarrow \infty$ , hence  $\dim_H \mathcal{I}_q \rightarrow 1$  and consequently  $\dim_H \mathcal{U}_q \rightarrow 1$ .

Observe that

$$(4.2) \quad \mathcal{I}_q = \bigcup S_{c_1 \dots c_N}(\mathcal{I}_q)$$

where the union runs over the words  $c_1 \dots c_N$  of length  $N$  consisting of zeros and ones satisfying  $0 < c_1 + \dots + c_N < N$ , and  $S_{c_1 \dots c_N} : J_q \rightarrow J_q$  is given by

$$S_{c_1 \dots c_N}(x) := \left( \frac{c_1}{q} + \dots + \frac{c_N}{q^N} \right) + \frac{x}{q^N}, \quad x \in J_q.$$

Moreover, the set  $\mathcal{I}_q$  is closed (and thus compact) because the limit of a monotonic sequence in  $\mathcal{I}_q$  converges to an element of  $\mathcal{I}_q$ . In other words,  $\mathcal{I}_q$  is the (nonempty compact) invariant set of the iterated function system formed by these  $2^N - 2$  similarities. The sets  $S_{c_1 \dots c_N}(\mathcal{I}_q)$  on the right side of (4.2) are disjoint because  $S_{c_1 \dots c_N}(\mathcal{I}_q) \subset \mathcal{I}_q \subset \mathcal{U}_q$ , and since all similarity ratios are equal to  $q^{-N}$ , it follows from Propositions 9.6 and 9.7 in [9] that the Hausdorff dimension  $s$  of  $\mathcal{I}_q$  is the real solution of the equation

$$(2^N - 2)q^{-Ns} = 1,$$

which is equivalent to (4.1).

(ii) Let  $q > 1$  be a non-integer and let  $n \in \mathbb{N}$  be such that  $\alpha_n(q) < \alpha_1(q)$ . It follows from Proposition 3.1 that the word  $1(0)^n$  does not occur in  $(b_i(x, q))$  if  $x$  belongs to  $\mathcal{U}_q$ . Applying Theorem 2.8(i) with  $y = q^{-1}$ ,  $\ell = 0$  and  $m = n + 1$ , we conclude that  $\dim_H \mathcal{U}_q < 1$ .  $\square$

*Proof of Theorem 1.1.* (ii) Let  $q > 1$  be a non-integer. Since  $\mathcal{V}_q \setminus \mathcal{U}_q$  is countable, Proposition 4.1(ii) yields that  $\dim_H \mathcal{V}_q < 1$ . This implies in particular that the set  $\mathcal{V}_q$  is a one-dimensional null set. Applying Theorem 1.2 (and the remark following its proof) and Fubini's theorem we conclude that  $\overline{\mathbf{U}}$  is a two-dimensional null set.

(i) Since  $\mathcal{U}_q$  is not closed for all  $q > 1$ ,  $\mathbf{U}$  cannot be closed. Since  $\overline{\mathbf{U}}$  is a two-dimensional null set, it has no interior points. It remains to show that  $\mathbf{U}$  (and thus  $\overline{\mathbf{U}}$ ) has no isolated points. If  $q > 1$  is an integer, then, as is well known,  $\mathcal{U}_q$  is dense in  $J_q = [0, 1]$ . If  $q > 1$  is a non-integer, then each  $(x, q) \in \mathbf{U}$  is not isolated because  $\mathcal{U}'_q \subset \mathcal{U}'_r$  whenever  $q < r$  and  $\lceil q \rceil = \lceil r \rceil$ .

(iii) From Corollary 7.10 in [9] we may conclude that for almost all  $q > 1$

$$\dim_H \mathcal{U}_q \leq \max \{0, \dim_H \mathbf{U} - 1\}$$

which would contradict Proposition 4.1(i) if we had  $\dim_H \mathbf{U} < 2$ .  $\square$

*Acknowledgements.* We warmly thank the anonymous referee for suggesting alternative proofs of Theorem 2.7(i) and Theorem 2.8(i) (see the last remark of Section 2), and for a very careful reading of the manuscript. The first author has been supported by NWO, Project nr. ISK04G. Part of this work was done during a visit of the second author at the Department of Mathematics of the Delft University of Technology. He is grateful for this invitation and for the excellent working conditions.

## REFERENCES

- [1] C. Baiocchi, V. Komornik, *Greedy and quasi-greedy expansions in non-integer bases*, arXiv:0710.3001 [math.], October 16, 2007.
- [2] K. Dajani, M. de Vries, *Invariant densities for random  $\beta$ -expansions*, J. Eur. Math. Soc. **9** (2007), no. 1, 157–176.
- [3] Z. Daróczy, I. Kátai, *Univoque sequences*, Publ. Math. Debrecen **42** (1993), no. 3–4, 397–407.
- [4] Z. Daróczy, I. Kátai, *On the structure of univoque numbers*, Publ. Math. Debrecen **46** (1995), no. 3–4, 385–408.
- [5] M. de Vries, V. Komornik, *Unique expansions of real numbers*, Adv. Math. **221** (2009), no. 2, 390–427.

- [6] P. Erdős, Horváth, M., I. Joó, *On the uniqueness of the expansions  $1 = \sum q^{-n_i}$* , Acta Math. Hungar. **58** (1991), no. 3–4, 333–342.
- [7] P. Erdős, I. Joó, V. Komornik, *Characterization of the unique expansions  $1 = \sum_{i=1}^{\infty} q^{-n_i}$  and related problems*, Bull. Soc. Math. France **118** (1990), no. 3, 377–390.
- [8] P. Erdős, I. Joó, V. Komornik, *On the number of  $q$ -expansions*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **37** (1994), 109–118.
- [9] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley & Sons, Chichester, second edition, 2003.
- [10] H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation*, Math. Sys. Theory **1** (1967), 1–49.
- [11] P. Glendinning, N. Sidorov, *Unique representations of real numbers in non-integer bases*, Math. Res. Lett. **8** (2001), no. 4, 535–543.
- [12] F. Hofbauer,  *$\beta$ -shifts have unique maximal measure*, Monatsh. Math. **85** (1978), no. 3, 189–198.
- [13] G. Kallós, *The structure of the univoque set in the small case*, Publ. Math. Debrecen **54** (1999), no. 1–2, 153–164.
- [14] G. Kallós, *The structure of the univoque set in the big case*, Publ. Math. Debrecen **59** (2001), no. 3–4, 471–489.
- [15] I. Kátai, G. Kallós, *On the set for which 1 is univoque*, Publ. Math. Debrecen **58** (2001), no. 4, 743–750.
- [16] V. Komornik, P. Loreti, *Unique developments in non-integer bases*, Amer. Math. Monthly **105** (1998), no. 7, 636–639.
- [17] V. Komornik, P. Loreti, *On the topological structure of univoque sets*, J. Number Theory, **122** (2007), no. 1, 157–183.
- [18] W. Parry, *On the  $\beta$ -expansions of real numbers*, Acta Math. Acad. Sci. Hungar. **11** (1960), 401–416.
- [19] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hungar. **8** (1957), 477–493.
- [20] N. Sidorov, *Almost every number has a continuum of  $\beta$ -expansions*, Amer. Math. Monthly **110** (2003), no. 9, 838–842.
- [21] N. Sidorov, *Combinatorics of linear iterated function systems with overlaps*, Nonlinearity **20** (2007), 1299–1312.

DELFT UNIVERSITY OF TECHNOLOGY, MEKELWEG 4, 2628 CD DELFT, THE NETHERLANDS  
*E-mail address:* `martijndevries0@gmail.com`

DÉPARTEMENT DE MATHÉMATIQUE, UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ DESCARTES,  
 67084 STRASBOURG CEDEX, FRANCE  
*E-mail address:* `vilmos.komornik@math.unistra.fr`